A JORDAN-HÖLDER THEOREM

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1. The purpose of this note is to present a certain general theorem, of the Jordan-Holder type, for finite groups. This theorem, although a simple and natural extension of the classical theorem, has we believe passed unnoticed before. The technique of proof is foreign to the usual methods of finite group theory, but seems well-suited to the situation.

2. A nonempty class $\mathscr{D}$ of finite groups will be called a *genetic class* provided:

(1) If G_1 belongs to $\mathscr{D}$ and if G_2 is isomorphic to G_1, then G_2 belongs to $\mathscr{D}$.

(2) If G belongs to $\mathscr{D}$, then every normal subgroup and every quotient group of G also belongs to $\mathscr{D}$.

The following examples of genetic classes will be used as illustrations in the sequel:

The class $\mathscr{G}$ of all finite groups.

The class $\mathscr{E}$ of all one-element groups.

The class $\mathscr{A}$ of all finite abelian groups.

The class $\mathscr{O}$ of all groups of odd order.

The class $\mathscr{G}_n$ of all groups of order $\leqq n$.

Given any genetic class $\mathscr{D}$, we shall construct a "Grothendieck group" in the following way. Let Σ be the (countable) set of all isomorphism classes of finite groups, and let F be the free abelian group generated by Σ. If G is any finite group, its isomorphism class will be denoted by $[G]$, so that elements of F are finite sums

$$\Sigma\lambda_i[G_i], \quad \lambda_i \in Z ,$$

where Z denotes the ring of integers. We let $N(\mathscr{D})$ be the subgroup of F generated by all elements of the form

$$[G] - [H] - [G/H]$$

such that H is a normal subgroup of G and G/H belongs to the genetic class $\mathscr{D}$. Finally we set $K(\mathscr{D}) = F/N(\mathscr{D})$ and let k: $F \to K(\mathscr{D})$ be the natural epimorphism. Our object is to determine the structure of the abelian group $K(\mathscr{D})$.

3. Let $\mathscr{D}$ be a genetic class and let G be an arbitrary finite

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