ON THE EXTENSIONS OF LATTICE-ORDERED GROUPS

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1. Introduction. Throughout this paper $A = 0, a, b, \cdots, \Delta = \theta, \alpha, \beta, \cdots$ and G will be abelian partially ordered groups (p.o. groups). G is a *p.o. extension* of A by Δ if there is an order preserving homomorphism (o-homomorphisn) π of G onto Δ with kernel A such that π induces an o-isomorphism of G/A with Δ, (i.e. $\pi(g) > \theta$ implies $g + A$ contains a positive element). If A and Δ are lattice ordered groups (l-groups) then G is an *l-extension* if G is an l-group, π is an l-homomorphism and π induces an l-isomorphism between G/A and Δ. In this case A is an l-ideal of G.

If G is a p.o. extension of A by Δ then for each $\alpha \in \Delta$ choose $r(\alpha) \in G$ such that $\pi(r(\alpha)) = \alpha$ and $r(\theta) = 0$. Define

$$f(\alpha, \beta) = -r(\alpha + \beta) + r(\alpha) + r(\beta) \quad \text{for all } \alpha, \beta \in \Delta$$

and

$$Q_\alpha = \{a \in A \mid r(\alpha) + a \geqq 0\} \quad \text{for } \alpha \in \Delta^+ = \{\delta \in \Delta \mid \delta \geqq \theta\} .$$

Then the following conditions are satisfied for all α, β, γ in Δ.

(i) $f(\alpha, \beta) = f(\beta, \alpha)$

(ii) $f(\alpha, \theta) = f(\theta, \alpha) = 0$

(iii) $f(\alpha, \beta) + f(\alpha + \beta, \gamma) = f(\alpha, \beta + \gamma) + f(\beta, \gamma)$.

Moreover, for $\alpha, \beta \in \Delta^+$ we have

(iv) $Q_\alpha \neq \phi$

(v) $Q_\alpha + Q_\beta + f(\alpha, \beta) \subseteqq Q_{\alpha+\beta}$

(vi) $Q_\theta = A^+$.

Conditions (iv)–(vi) are due to L. Fuchs and can be derived from the results in [5].

Now if $\bar{G} = A \times \Delta$ and we define $(a, \alpha) + (b, \beta) = (a + b + f(\alpha, \beta), \alpha + \beta)$ and (a, α) positive if $\alpha \in \Delta^+$ and $a \in Q_\alpha$, then the mapping $(a, \alpha) \rightarrow r(\alpha) + a$ is an o-isomorphism of $\bar{G}$ onto G. In what follows we usually identify G and $\bar{G}$.

Conversely, if we are given $A, \Delta, f\colon \Delta \times \Delta \rightarrow A$ and $Q\colon \Delta^+ \rightarrow$ {subsets of A} such that f and Q satisfy (i)–(vi) then $\bar{G}$ is a p.o. extension of A by Δ and the mapping $(a, \alpha) \rightarrow \alpha$ is the corresponding o-homomorphism.

Two p.o. extensions $G = (A, \Delta, f, Q)$ and $G' = (A, \Delta, f', Q')$ are *o-equivalent* if there is a function $t\colon \Delta \rightarrow A$ such that

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