WEAKLY COMPACT OPERATORS ON OPERATOR ALGEBRAS

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Let K be a compact space and $C(K)$ be the commutative B^*-algebra of all complex valued continuous functions on K, then Grothendieck [3] (also we can see other proofs in [2]) proved the following remarkable properties:

(I) An arbitrary bounded operator of $C(K)$ into a weakly sequentially complete Banach space is weakly compact.

(II) If T is a weakly compact operator of $C(K)$ into a Banach space, then T maps weakly fundamental sequences into strongly convergent sequences.

On the other hand, let M be a W^*-algebra and M_* be the associated space of M (namely, the dual of M_* is M (cf. [8])) then the author [7] noticed that the Banach space M_* is weakly sequentially complete. Therefore, the above Grothendieck's theorems are applicable in the theory of operator algebras.

In this note, we shall show some applications, and state some related problems.

PROPOSITION 1. Let A be a B^*-algebra, E an abstract L-space, T be a bounded operator of A into E, then T is weakly compact.

Proof. Let T^* be the dual of T, then T^* is a bounded operator on the dual E^* of E to the dual A^* of A; E^* is a Banach space of type $C(K)$ (cf. [5]) and the second dual A^{**} of A is a W^*-algebra (cf. [9]), so that A^* is the associated space of a W^*-algebra; hence A^* is weakly sequentially complete; therefore T^* is weakly compact, so that by the well-known theorem, T is weakly compact. This completes the proof.

Now we shall show some applications.

1. Let G be a locally compact group, $L^1(G)$ be the Banach space of all complex valued integrable functions on G with respect to a left, invariant Haar measure μ and $L^2(G)$ be the Banach space of all complex valued square integrable functions on G with respect to μ. Under the convolutions (denoted by "$*$"), $L^1(G)$ is a Banach algebra.

On the other hand, for $f \in L^1(G)$ and $g \in L^2(G)$, put $L_f g = f * g$,

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