TOEPLITZ MATRICES AND INVERTIBILITY OF HANKEL MATRICES

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1. Introduction. Let $\{c_n\}$, for $n = 0, \pm 1, \pm 2, \cdots$, be a sequence of real numbers satisfying $c_0 = 0, c_{-n} = c_n$ and $0 < \sum_{n=1}^{\infty} c_n^2 < \infty$, and let $f(\theta)$ ($\not\equiv 0$) be the even function of class $L^2(-\pi, \pi)$ defined by

$$(1) \qquad f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = 2 \sum_{n=1}^{\infty} c_n \cos n\theta \, .$$

Define the Toeplitz matrix T and the Hankel matrices H and K by

$$(2) \quad T = (c_{i-j}),\ H = (c_{i+j-1}) \text{ and } K = (c_{i+j}), \quad \text{where } i, j = 1, 2, \cdots .$$

Then

$$(3) \qquad T = F + K\,, \quad \text{where } F = \int_0^{\pi} f(\theta) dE_0(\theta)\,,$$

and $\{E_0(\theta)\}$ is the resolution of the identity of the matrix belonging to the quadratic form $2 \sum_{n=1}^{\infty} x_n x_{n+1}$. (See [12], p. 837.)

A self-adjoint operator A on a Hilbert space, with the spectral resolution $A = \int \lambda dE(\lambda)$, will be called absolutely continuous if $\| E(\lambda)x \|^2$ is an absolutely continuous function of λ for every element x of the Hilbert space. If the function $f(\theta)$ of (1) is (essentially) bounded then T must be bounded (Toeplitz). Since F must also be bounded, so also are H and K. It was shown in [12], p. 840, using methods involving commutators of operators, that if the function $g(\theta)$ defined by

$$(4) \qquad g(\theta) \sim \sum_{n=1}^{\infty} c_n e^{in\theta}$$

is bounded (hence $f(\theta)$ is also bounded) then T must be absolutely continuous if either

(5) 0 is not in the point spectrum of H (that is, H^{-1} exists),

or

(6) F is absolutely continuous.

Rosenblum [17] has shown, using results of Aronszajn and Donoghue [1], that in fact T is *always* (with no restrictions) absolutely continuous.

Received February 8, 1963. This research was supported by the National Science Foundation research grant NSF-G18915.