B^* ALGEBRA UNIT BALL EXTREMAL POINTS

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Results of Kadison [3] and Jacobson [2] are combined to show that the points described by the title are unitaries, left shifts, right shifts, or sums of these. The extremality property is preserved by homomorphisms; conversely, when range and domain are AW^* algebras, every extremal point of the range has an extremal point in its pre-image. Exact formulations of these results and of a few simple consequences are given in section one; proofs follow in section two.

In what follows, A will be a self-adjoint subalgebra of some B^* algebra; "x is extremal (A)" will mean that x is an extremal point of the unit ball of A with respect to the B^* norm indicated by the context; "weak topology" will mean the weak operator topology with respect to the representation of A by bounded operators on a Hilbert space which is indicated by the context.

1. Theorems. Our starting point is a formula due to Kadison ([3], Theorem 1). In a mildly generalized form, his result is:

THEOREM 1. *Let A be a self-adjoint subalgebra of some B^* algebra B. Then x is extremal (A) if and only if*

$$(1 - x^*x)A(1 - xx^*) = \{0\} .$$

Here "1" stands for the identity of A if there is one; otherwise the meaning of the equation is to be found by performing the indicated multiplications for each $y \in A$. It turns out (Theorem 2) that the existence of any element extremal (A) implies that A has an identity.[1]

An obvious consequence of this formula is the perseverance of extremality. Calling "reasonable" any linear topology making involution continuous, and multiplication continuous in each variable separately, we have:

COROLLARY (i) *If $\bar{A}$ is the closure of A in B with respect to a reasonable topology, and if x is in A, then x is extremal (A) if and only if x is extremal $(\bar{A})$.*

(ii) *If ϕ is a $*$-homomorphism of A into a B^* algebra B_1, then x extremal (A) implies that ϕx is extremal (ϕA).*

Using the methods of [2], one can draw substantial information about the form of an individual extremal element from Theorem 1.

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[1] This has already been proved by Sakai [5, p. 1.3]