ON THE DIOPHANTINE EQUATION $Cx^2 + D = y^n$

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1. Introduction. Let C, D and n denote odd positive integers, $D > 1$ and CD without any squared factor > 1. Let $K = Q(\sqrt{-CD})$, where Q is the field of rational numbers. Let further h denote the number of classes of ideals in K and put $D + (-1)^{(D+1)/2} = 2^m \cdot D_1$, $(D_1, 2) = 1$. In two previous papers [4] and [5] I have proved the following three theorems concerning the diophantine equation $Cx^2 + D = y^n$:

I. The diophantine equation

$$(1) \qquad Cx^2 + D = y^n \,, \qquad n > 1$$

is impossible in rational integers x and y if $h \not\equiv 0 \pmod{n}$, m is odd and either $CD \equiv 1 \pmod 4$ or $CD \equiv 3 \pmod 8$ with $n \not\equiv 0 \pmod 3$.

II. The diophantine equation

$$(2) \qquad Cx^2 + D = y^q \,, \qquad q > 3$$

where q denote an odd prime and $CD \not\equiv 7 \pmod 8$, is impossible in rational integers x and y if $h \not\equiv 0 \pmod{q}$, m is even and $q \not\equiv CD_1 \pmod 8$.

III. If $D \equiv 1 \pmod 4$, $CD \not\equiv 7 \pmod 8$ and m is even, then the equation (2) has only a finite number of solutions in natural numbers x, y and primes q if $CD_1 \equiv 5 \pmod 8$ or if $C = 1$ with $D_1 \equiv 3 \pmod 8$ for given C and D. The possible values of y and an upper limit for the number of primes q may always be determined after a finite number of arithmetical operations.

From the proofs it immediately follows that these theorems also hold good if $CD \equiv 7 \pmod 8$, *provided y is an odd integer*. This gives a far-reaching extension of results obtained by D. J. Lewis in his paper [2]. Putting $C = 1$, $D = 7$ we find, from 1:

The diophantine equation $x^2 + 7 = y^z$, $z > 1$, is impossible in rational integers x, y and z if y is an odd integer.

Equations of the type (1) have also been studied by T. Nagell [6], [8], [9] and B. Stolt [11].

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