ALGEBRAIC EXTENSIONS OF COMMUTATIVE BANACH ALGEBRAS

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1. Introduction. Let A denote a commutative normed algebra with multiplicative unit and norm $\|\cdot\|$. In [2], Arens and Hoffman showed that it is possible to norm $A[x]/(\alpha(x))$, where $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i$ is a monic polynomial over A, in such a way that the canonical mapping of A into $A[x]/(\alpha(x))$ is an isometry as well as an isomorphism; in fact, they give a family of norms on $A[x]/(\alpha())$, all of which are equivalent. Specifically, let t be a positive number which satisfies $t^n \geqq \|\alpha_0\| + \|\alpha_1\| t + \cdots + \|\alpha_{n-1}\| t^{n-1}$. Let $\sum_{i=0}^{n-1} a_i x^i + (\alpha(x))$ be any coset in $A[x]/(\alpha(x))$. As is well known, $\sum_{i=0}^{n-1} a_i x^i$ is the unique representative of this coset of lowest degree. Thus, $\| \sum_{i=0}^{n-1} a_i x^i + (\alpha(x)) \| = \sum_{i=0}^{n-1} \|a_i\| t^i$ is well defined and makes $A[x]/(\alpha(x))$ into a normed algebra. Clearly, $a \rightarrow a + (\alpha(x)), a \in A$, is an isometry of A into $A[x]/(\alpha(x))$. (Unless otherwise stated, we assume without loss of generality that $t = 1$.) From the form of the norm we see that $A[x]/(\alpha(x))$ is a Banach algebra under this norm precisely when A is a Banach algebra under $\|\cdot\|$. In the present paper, we deal mainly with the case where A is a Banach algebra. In section nine we deal with, at some length, more general algebras.

In this paper we are mainly interested in the algebraic aspects of the extension $B = A[x]/(\alpha(x))$. However, we also present results which are Banach algebraic in nature. For example in section three we give a complete description of the Šilov boundary of B. Section four is devoted to the study of the inheritance by B of the Banach algebra properties of regularity and self-adjointness. In particular, we show that if A is regular then B is also regular. Self-adjointness is not always inherited as Example 4.3 shows. A sufficient condition (which is satisfied, for example, when the discriminant of $\alpha(x)$ is invertible) is given under which this property is inherited. (This condition states that the set $S(\alpha(x), A)$ of singular points of $\alpha(x)$ is empty. This means that the natural mapping of the carrier space of B onto the carrier space of A is a local homeomorphism with respect to the weak* topologies. See section two for a complete discussion of this concept.)

In section five we once again make use of the condition that $\alpha(x)$ has no singular points. Theorem 5.2 states that if A is semi-simple

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