FIELDS DEFINED BY POLYNOMIALS

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1. Introduction. First we consider the following question, where F is any field. For what pairs P and Q of polynomials in two variables with coefficients in F do the definitions

$$\text{(I)} \qquad a \oplus b = P(a, b)\ , \qquad a \odot b = Q(a, b)\ ,$$

for all a and b in F yield a field $(F, \oplus, \odot)$? It turns out that the answer is different for infinite fields than for finite fields, as shown in §§ 2 and 3.

Next let R be the field of real numbers. For what quadruples P_1, P_2, Q_1, Q_2 of real polynomials in four variables is $(R \times R, \oplus, \odot)$ a field, when we set

$$\text{(II)} \qquad \begin{aligned} (a, b) \oplus (c, d) &= (P_1(a, b, c, d), P_2(a, b, c, d))\ , \\ (a, b) \odot (c, d) &= (Q_1(a, b, c, d), Q_2(a, b, c, d))\ , \end{aligned}$$

where (x, y) denotes an ordered pair of real numbers? This question is partially answered in §§ 4 and 5, and in § 6 it is shown that the polynomials may be of arbitrarily high degree. In § 7 it is proved that if definitions (II) do give a field, it must be isomorphic to the field of complex numbers.

2. The one-dimensional case.

THEOREM 1. *Let F be an infinite field. The system $(F, \oplus, \odot)$ in* (I) *is a field if and only if*

$$(1) \qquad \begin{aligned} P(a, b) &= a \oplus b = a + b + \gamma \\ Q(a, b) &= a \odot b = \gamma\sigma(a + b) + \sigma ab + \gamma^2\sigma - \gamma\ , \end{aligned}$$

where $\gamma \in F$, $\sigma \in F$ and $\sigma \neq 0$. When these conditions are satisfied the field $(F, \oplus, \odot)$ is isomorphic to F, thus $(F, \oplus, \odot) \cong (F, +, \cdot)$.

Proof. We first assume that $(F, \oplus, \odot)$ is a field and show that the polynomials P and Q have the prescribed form. By associativity we have $P(P(a, b), c) = P(a, P(b, c))$ identically in a, b, c. Now if P is of degree n in a, the degrees of the left and right sides of this identity in a are n^2 and n respectively. Since F is infinite it follows

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