DOUBLY INVARIANT SUBSPACES, II

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1. Introduction. Let X be a locally compact Hausdorff space and μ a positive Radon measure on X. Let $\mathscr{H}$ be a separable Hibert space and let $L^p_{\mathscr{H}}$ $(1 \leqq p \leqq +\infty)$ denote the space of $\mathscr{H}$-valued functions on X which are weakly measurable and whose norms are in scalar $L^p(d\mu)$. Call P a *measurable range function* if P is a function on X defined a.e. $(d\mu)$ to the space of orthogonal projections on $\mathscr{H}$ which is weakly measurable. We shall regard two range functions P, P' to be the same if $P(x) = P'(x)$ l.a.e., i.e. $P(x) = P'(x)$ a.e. on every compact subset of X. We shall denote by $\hat{P}$ the operator on $L^p_{\mathscr{H}}$ defined by $(\hat{P}f)(x) = P(x)f(x)$ l.a.e. Let A be a subalgebra of the algebra $C(X)$ of bounded continuous functions on X such that $A \cup \bar{A}$ (where the bar denotes complex conjugation) is weakly* dense in $L^\infty(d\mu)$. Say that a subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ is *doubly invariant* if

(i) $\mathscr{M}$ is closed in $L^p_{\mathscr{H}}$ if $1 \leqq p < \infty$ and weakly* closed if $p = \infty$,

(ii) $\mathscr{M}$ is invariant under multiplication by functions in $A \cup \bar{A}$.

We shall refer to the following theorem as Wiener's theorem for $L^p_{\mathscr{H}}$:

THEOREM. *Every doubly invariant subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ $(1 \leqq p \leqq \infty)$ is of the form $\hat{P}L^p_{\mathscr{H}}$ for some measurable range function P (and trivially conversely); $\mathscr{M}$ determines P uniquely.*

For compact spaces X, Wiener's theorem was proved in [4] for arbitrary $\mathscr{H}$ for $p = 2$ and for the scalar $\mathscr{H}$ (the space of complex numbers) for arbitrary p. It was pointed out in [4] that the $L^2_{\mathscr{H}}$ theorem is true for locally compact spaces and the proof was outlined considering the real line as an example. It was also mentioned in [4] that the $L^2_{\mathscr{H}}$ theorem is a special case of a known theorem on rings of operators [2; p. 167, Théorème 1]. But the proof in [4] and the proof of the more general theorem in [2] implicitly assume the σ-finiteness of μ or at least of the separability of $L^2_{\mathscr{H}}$ (as opposed to the separability of $\mathscr{H}$). The theorem itself is true without this restriction not only for $p = 2$ but for all p and all (separable) $\mathscr{H}$ (not necessarily the scalar $\mathscr{H}$). Indeed the general $L^p_{\mathscr{H}}$ theorem is true even under the weaker assumption that the restriction of $A \cup \bar{A}$ to every compact subset K of X is L^2-dense in $L^2(d\mu \mid K)$, instead of being weakly* dense in L^∞. In this paper we prove this theorem

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