INTRINSIC EXTENSIONS OF RINGS

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A module M is an *essential extension* of a submodule N in case $K \cap N \neq 0$ for each nonzero submodule K of M. If S is a subring of a ring R, and if ${}_SR$, ${}_SS$ denote the left S-modules naturally defined by the ring operations of R, then R is a *left quotient ring of* S in case ${}_SR$ is an essential extension of ${}_SS$.

We shall discuss the following problem: (1) Characterize the condition that a ring extension R of S is a left quotient ring of S wholly in terms of the relative left ideal structures of R and S.

A ring extension R of S is *left intrinsic over* S in case $K \cap S \neq 0$ for each nonzero left ideal K of R. Evidently each left quotient ring R of S is left intrinsic over S but an obvious example (when R is a field and S a subfield $\neq R$) shows that the converse fails. Nevertheless, we ask: (2) When is the condition R is left intrinsic over S a solution to (1)?

We now specialize S by requiring that:

(i) S possesses a left quotient ring which is a (von Neumann) regular ring, or equivalently (R. E. Johnson [2]) by requiring that the left singular ideal of S vanishes. For such a ring there exists a maximal left quotient ring $\hat{S}$ which is unique up to isomorphism over S, and which is itself a regular ring ([2]). To eliminate the field example we require that:

(ii) $\hat{S}$ possesses no strongly regular ideals $\neq 0$. Under these hypotheses we present the following solution to (1).

A. THEOREM (2.6). *Let S satisfy* (i) *and* (ii). *Then an extension ring R of S is a left quotient ring of S if and only if R is a left intrinsic extension of S such that for each closed left ideal A of S there corresponds a left ideal B of R such that $B \cap S = A$.*

(See § 1 for definitions.)

Regarding (2) we add a rather dubious final hypothesis:

(iii) $\hat{S}$ is right intrinsic over S.

B. THEOREM (3.1). *If S satisfies* (i)–(iii), *then an extension ring R of S is a left quotient ring of S if and only if R is left*

Received May 15, 1963. The first author gratefully acknowledges partial support from the National Science Foundation under grants G-19863 and G-21514.