CHARACTERIZATIONS OF CONVOLUTION SEMIGROUPS OF MEASURES

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A problem of fundamental importance in the study of compact topological semigroups is that of classifying in an *intrinsic* way each of a certain class of such semigroups. Unfortunately, virtually nothing has been done along these lines, even for such geometrically pleasing semigroups as the affine semigroups introduced by the author and H. Cohen in [3]. It is the purpose of this note to rectify this situation, at least for several particular types of compact affine topological semigroups; namely, certain convolution semigroups of real valued regular Borel measures on compact topological semigroups. The author's interest in this problem dates back to the early papers of Peck [13] and Wendel [21], and to some unpublished work of Wendel. Since that time, quite a literature has developed as regards these semigroups (e.g., see the bibliography), but almost without exception these papers merely study the *properties* of the semigroups without making any attempt to abstract sufficiently many of their properties to characterize them.

If S is a compact Hausdorff space and $P(S)$ denotes the set of all nonnegative regular Borel measures on S of variation norm one, it is known that $P(S)$ is a convex set which is compact in the weak-* topology $\left(\text{a net } \{\mu a\} \text{ of measures in } P(S) \text{ converges weak-* to } \mu \in P(S) \text{ if } \int f d\mu a \to \int f d\mu, \text{ for each real continuous function } f \text{ on } S\right)$. In similar fashion, the unit ball $B(S)$ of real-valued regular Borel measures of norm ≤ 1 is a compact convex set. When S is endowed with a continuous associative multiplication, each of $P(S)$ and $B(S)$ becomes a compact affine topological semigroup relative to convolution multiplication (see [10]); when such is the case, we denote these semigroups by $\tilde{S}$ and $\tilde{\tilde{S}}$ respectively. Note that our use of the symbol $\tilde{\tilde{S}}$ differs from that of Glicksberg in [10], where $\tilde{\tilde{S}}$, denoted the ball semigroup of *complex* measures.

In § 2, the following three types of images of the sets $P(S)$ and $B(S)$ are determined:

(a) all *extremal* images of $P(S)$; i.e., all continuous affine images under mappings which preserve extreme points,

(b) all one-to-one affine bicontinuous images of $P(S)$, and

(c) all one-to-one affine bicontinuous images of $B(S)$. The common requirements in each of (a), (b), and (c) are that the image K be

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