LENGTH-PRESERVING MAPS

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1. Introduction. If any two points of the metric space R can be connected by a rectifiable curve then a map of R into a metric space R' is *length-preserving or equilong*, if the length of any curve in R equals that of its image in R'. An equilong map of R means such a map of R into itself.

Folding a piece of paper repeatedly and in different ways exhibits a great variety of equilong maps of the euclidean plane. The original purpose of the present investigation was to determine all equilong maps which are not too pathological of the euclidean spaces and to find out whether other interesting[1] spaces admit length preserving maps which are not isometries.

However, equilong maps are connected with other important concepts. If the metric of R is intrinsic, i.e., if the distance of any two points equals the infimum of the lengths of all curves in R connecting these points, then an equilong map α of R into a metric space R' does not increase distance: $xy \geqq \alpha x \alpha y$. We denote as *shrinkage* any map of a metric space R into R' satisfying this inequality. Shrinkages which are not equilong enter significantly many branches of mathematics.[2] In fact, the linguistically preferable term "contraction" was avoided here, because it is widely used in functional analysis for the special shrinkages satisfying $xy \geqq k\alpha x \alpha y$ with $k > 1$ (see, for example, [5]). Therefore it seemed worthwhile to study the elementary properties of shrinkages as such.

On the other hand, isometries and local isometries are most important special maps (the latter in the theory of covering spaces) which preserve length. Our results on equilong maps will allow us to weaken the hypotheses in various theorems concerning (local) isometries. It often turns out that the axioms for a G-space (see [1]) need not all be satisfied and that a map can be proved to be onto where hitherto this had been assumed.

As to our original aims: we will *determine all locally finite equilong maps of the euclidean, hyperbolic, and spherical spaces.* The maximal open connected sets on which an equilong map is in-

Received May 22, 1963. This work was supported by a grant from the National Science Foundation.

[1] "Interesting" is an essential qualification because there are many spaces with isolated equilong maps.

[2] Among the less known applications, the shrinkages of cones on certain surfaces constructed by Rešetnyak [**6**] deserve special mention, because they yield elegant solutions of extremal problems in differential geometry.