ON THE FUNCTIONAL EQUATION $F(mn)F((m, n)) = F(m)F(n)f((m, n))$

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1. Introduction. Let f be a completely multiplicative arithmetical function. That is, f is a complex-valued function defined on the positive integers such that

$$f(mn) = f(m)f(n)$$

for all m and n. We allow the possibility that $f(n) = 0$ for all n. (If f is not identically zero then we must have $f(1) = 1$.) Given such an f we wish to study the problem of characterizing all numerical functions F which satisfy the functional equation

$$F(mn)F((m, n)) = F(m)F(n)f((m, n)) , \tag{1}$$

where (m, n) denotes the greatest common divisor of m and n. When $f(n) = n$ for all n, Equation (1) is satisfied by the Euler ϕ function since we have

$$\phi(mn)\phi((m, n)) = \phi(m)\phi(n)(m, n) .$$

More generally, it is known (see [1], [2]) that an infinite class of solutions of (1) is given by the formula

$$F(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right)g\left(\frac{n}{d}\right) ,$$

where μ is the Möbius function and g is any multiplicative function, that is,

$$g(mn) = g(m)g(n) \quad \text{whenever } (m, n) = 1 .$$

Some work on a special case of this problem has been done by P. Comment [2]. In the case $f(1) = 1$ he has investigated those solutions F of (1) which have $F(1) \neq 0$ and which satisfy an additional condition which he calls "property O": If there exists a prime p_0 such that $F(p_0) = 0$ then $F(p_0^\alpha) = 0$ for all $\alpha > 1$. Comment's principal theorem states that F is a solution of (1) with property O and with $F(1) \neq 0$ if, and only if, F satisfies the two equations

$$F(mn)F(1) = F(m)F(n) \quad \text{whenever } (m, n) = 1$$

and

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