LINEAR TRANSFORMATIONS ON GRASSMANN SPACES

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1. Let U denote an n-dimensional vector space over an algebraically closed field F, and let G_{nr} denote the set of nonzero pure r-vectors of the Grassmann product space $\bigwedge^r U$. Let T be a linear transformation of $\bigwedge^r U$ which sends G_{nr} into G_{nr}. In this note we prove that T is nonsingular, and then, by using the results of Wei-Liang Chow in [1], we determine the structure of T.

For each $z = x_1 \wedge \cdots \wedge x_r \in G_{nr}$, we let $[z]$ denote the r-dimensional subspace of U spanned by the vectors $x_1, \cdots, x_r$. By Lemma 5 of [1], two independent elements z_1 and z_2 of G_{nr} span a subspace all of whose nonzero elements are in G_{nr} if and only if $\dim([z_1] \cap [z_2]) = r - 1$; that is, if and only if $[z_1]$ and $[z_2]$ are adjacent. If $V \subseteqq \bigwedge^r U$ is a subspace such that each nonzero vector in V is in G_{nr} and if V is maximal (that is, not contained in a larger such subspace) then $\{[z] \mid z \in V, z \neq 0\}$ is a maximal set of pairwise adjacent r-dimensional subspaces of U. These sets of subspaces are of two types; namely, the set of all r-dimensional subspaces of U containing a common $(r-1)$-dimensional subspace, and the set of all r-dimensional subspaces of an $(r+1)$-dimensional subspace of U. We adopt the usual convention of calling these sets of subspaces maximal sets of the first and second kind respectively. We will let A_r denote the set of those maximal V which determine a set of pairwise adjacint subspaces of the first kind, and we will let B_r denote the set of those maximal V which determine a set of pairwise adjacent subspaces of the second kind.

2. In this section we prove that if T sends each member of B_r into a member of B_r then T is nonsingular.

Let $U_1, \cdots, U_t$ be k-dimensional pairwise adjacent subspaces of U and let $z_i \in G_{nk}$ be such that $[z_i] = U_i$ for $i = 1, \cdots, t$. Then $\{U_1, \cdots, U_t\}$ is said to be independent if and only if $\{z_1, \cdots, z_t\}$ is an independent subset of $\bigwedge^k U$. We note the following facts concerning an independent set $\{U_1, \cdots, U_t\}$. If it is of the first kind (in the sense of the previous section) then there is an independent set of vectors $\{x_1, \cdots, x_{k-1}, y_1, \cdots, y_t\}$ of U such that for $i = 1, \cdots, t$, $U_i = \langle x_1, \cdots, x_{k-1}, y_i\rangle \cdot \langle \cdots \rangle$ denotes the linear subspace spanned by the vectors enclosed. If it is of the second kind, then there is an independent set of vectors $\{x_1, \cdots, x_{k+1}\}$ such that $U_i = \langle x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}\rangle$, for $i = 1, \cdots, t$. It is easily

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