# ON THE LOCATION OF THE ZEROS OF SOME INFRAPOLYNOMIALS WITH PRESCRIBED COEFFICIENTS

J. L. WALSH AND O. SHISHA

1. Various results have been obtained regarding the zeros of infrapolynomials with prescribed coefficients. (See e.g. [Walsh, 1958], [Walsh and Zedek, 1956], [Fekete and Walsh, 1957], [Shisha and Walsh, 1961, 1963], and [Shisha, 1962]). Our purpose in the present note is twofold:

(i) to contribute more deeply to that study, making use of some properties of polynomials and rational functions, and

(ii) conversely, further to show how results concerning infrapolynomials can be used in the investigation of some rational functions and in particular some combinations of a polynomial and its derivative.

2. We repeat here the underlying definition. Let $n$ and $q$ be natural numbers ($q \leqq n$), $n_1, n_2, \cdots, n_q$ integers such that $0 \leqq n_1 < n_2 \cdots < n_q \leqq n$, and $S$ a pointset in the (open) complex plane. An $n$th infrapolynomial on $S$ with respect to $(n_1, n_2, \cdots, n_q)$ is a polynomial $A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$ having the property: There does not exist a polynomial $B(z) \equiv \sum_{\nu=0}^{n} b_\nu z^\nu$ such that $B(z) \not\equiv A(z)$, $b_{n_\nu} = a_{n_\nu}$ $(\nu = 1, 2, \cdots, q)$, $|B(z)| < |A(z)|$ whenever $z \in S$ and $A(z) \neq 0$, and $B(z) = 0$ whenever $z \in S$ and $A(z) = 0$.

3. Of special importance among the above sequences $(n_1, n_2, \cdots, n_q)$, are "simple $n$-sequences" [Shisha and Walsh, 1961]. Given a natural number $n$, we define a "simple $n$-sequence" to be a sequence having one of the forms $(0, 1, \cdots, k, n-l, n-l+1, \cdots, n)$ $[k \geqq 0, l \geqq 0, k + l + 2 \leqq n]$; $(0, 1, \cdots, k)$ $[0 \leqq k < n]$; $(n-l, n-l+1, \cdots, n)$ $[0 \leqq l < n]$. We shall consider $n$th infrapolynomials on some special sets $S$ with respect to simple $n$-sequences $\sigma$. The sets $S$ will consist of $n - s + 2$ points, where $s$ is the number of elements of $\sigma$, and $S$ will be required not to contain the origin, in case $\sigma$ contains zero. As explained in the Introduction to the last mentioned paper, this particular situation is of special importance, as the general case is to a large extent reducible to it, and as these particular $n$th infrapolynomials are closely related to certain combinations of a polynomial and its derivative. Numerous results on such combinations exist in the literature.

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