HOMOGENEOUS QUASIGROUPS

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A mathematical system whose group of automorphisms is transitive we will call homogeneous. If the group of automorphisms is doubly transitive, then we will call the system doubly homogeneous. We examine here homogeneous and doubly homogeneous finite quasigroups.

We prove that there are no homogeneous quasigroups whose order is twice an odd number (Theorem 1.1). As the quasigroups satisfying the identity $X(YZ) = XY \cdot XZ$ show, there are homogeneous quasigroups of all other orders ([5], p. 236).

We then examine doubly homogeneous quasigroups and show that they are intimately connected with nearfields (Theorem 2.2). Since all finite nearfields are known, we thus have a complete description of the doubly homogeneous quasigroups.

In the last two sections we obtain various equivalent descriptions of double homogeneity and apply them to the construction of block designs and models for certain identities.

1. Homogeneous quasigroups.

In this section two theorems are obtained that generalize results concerning distributive quasigroups.

THEOREM 1.1. *There is no homogeneous quasigroup of order* $4k + 2$.

Proof. Let $(Q, \circ)$ be a homogeneous quasigroup of order $4k + 2$. We first construct out of this quasigroup an idempotent homogeneous quasigroup of order $4k + 2$.

Define $f: Q \to Q$ by $f(x) = x \circ x$. We assert that f is onto Q, and hence a bijection. Indeed, let a be a fixed element of Q, $b = a \circ a$, c an arbitrary element of Q, g an automorphism of $(Q, \circ)$ such that $g(b) = c$. We then have

$$c = g(b) = g(a \circ a) = g(a) \circ g(a) = f(g(a)) .$$

Thus f is onto Q.

We thus can define a quasigroup $(Q, \odot)$, isotopic to $(Q, \circ)$, by $f(x) \odot f(y) = x \circ y$. Since $f(x) \odot f(x) = x \circ x = f(x)$, $(Q, \odot)$ is idempotent. Moreover, if g is an automorphism of $(Q, \circ)$, it is also an automorphism of $(Q, \odot)$, since

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