# A PERMANENT INEQUALITY FOR POSITIVE FUNCTIONS ON THE UNIT SQUARE

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**Introduction.** During the past few years the van der Waerden conjecture on the minimum of the permanent of a doubly stochastic matrix has received considerable attention. (See Marcus and Newman [1] and [2], Marcus and Minc [1], among others.) This conjecture states that if $A$ is a doubly stochastic matrix, i.e. if

$$a_{ij} \geqq 0, \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1 ,$$

then the permanent of $A$ is $\geqq n!\, n^{-n}$. (The permanent of $A$ is $\sum \prod a_{i\sigma(i)}$, where the summation is taken over all permutations $\sigma$ in the symmetric group.) Despite the seemingly elementary character of the conjecture, it is, so far as the present authors are aware, still unresolved in general, although it has been settled in some special cases. (See the above references.)

An implication of the conjecture is that some term of the permanent expansion must be greater than or equal to $n^{-n}$. This was established by Marcus and Minc [1] in 1962. Specifically they showed that if $\prod a_{ii}$ is not exceeded by any other term in the permanent expansion, then

$$(1) \qquad \sum \log a_{ii} \geqq \sum \sum a_{ij} \log a_{ij} \geqq \mathrm{n} \log n^{-1} .$$

The second inequality above is a simple application of Jensen's inequality using the convex function $x \log x$; the first inequality is the key to the problem. It is the extension of this inequality to functions defined on the unit square that is referred to in the title of this paper. We will show in § 4 that under suitable hypotheses

$$(2) \qquad \infty > \int_0^1 \log f(x, x)\, dx \geqq \int_0^1 \int_0^1 f(x, y) \log f(x, y)\, dxdy \geqq 0 .$$

The proof of (2) (and incidentally a new proof of (1)) is based ultimately on the following theorem:

THEOREM 1. *Let $S$ be an arbitrary set and $f(p, q)$ a real-valued function defined on $S \times S$ with the following property:*

(C) *if $p_1, \cdots, p_n$ is any finite sequence of points in $S$, not necessarily distinct, then*

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