A CHARACTERIZATION OF WEAK* CONVERGENCE

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1. Introduction. Let X be a locally compact, Hausdorff space and $\{\mu_i\,; i \in D\}$ be a net of Radon measures on X (in the sense of Caratheodory). The weak* or vague limit of this net is the Radon measure ν such that

$$\lim_i \int f d\mu_i = \int f d\nu$$

for every continuous function f vanishing outside some compact set. In this paper, we construct in § 3 a Radon measure φ^* from a given base $\mathscr{B}$ for the topology of X and $\liminf_i \mu_i$ and then, in § 4, we give necessary and sufficient conditions for φ^* to be the weak* limit of the μ_i. In particular, if the latter exists then it is the φ^* generated when $\mathscr{B}$ is the family of all open sets.

The measure φ^* is obtained from another measure φ by a standard regularizing process. The definition of φ easily extends to abstract spaces but that of φ^* makes essential use of the topology. Thus, it is of some importance to know when $\varphi = \varphi^*$, that is, when a measure constructed through an abstract process from the μ_i turns out to be, in the topological situation, the weak* limit of the μ_i. In Theorem 3.3 we give a condition for $\varphi = \varphi^*$ and in § 5 we give an example to show that the condition cannot be eliminated.

We refer to standard texts such as Halmos [1], Kelley [2], and Munroe [3] for the elementary properties and concepts of topology and measure theory used in this paper.

2. Notation.

2.1 ω denotes the set of natural numbers.

2.2 0 denotes both the empty set and the smallest number in ω.

2.3 μ is a Caratheodory (outer) measure on X if and only if μ is a function on the family of all subsets of X such that $\mu 0 = 0$ and

$$0 \leqq \mu A \leqq \sum_{n \in \omega} \mu B_n \leqq \infty \quad \text{whenever } A \subset \bigcup_{n \in \omega} B_n \subset X .$$

2.4 For μ *a* Caratheodory measure on X, A is μ-measurable if and only if $A \subset X$ and for evey $T \subset X$

$$\mu T = \mu(T \cap A) + \mu(T - A) .$$

2.5 For X a topological space, μ is a Radon measure on X if and

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