QUASI-POSITIVE OPERATORS

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1. Introduction. The classical results of Perron and Frobenius ([6], [7], [12]) assert that a finite dimensional, nonnegative, non-nilpotent matrix has a positive eigenvalue which is not exceeded in absolute value by any other eigenvalue and the matrix has a nonnegative eigenvector corresponding to this positive eigenvalue. If the matrix has strictly positive entries, then there is a positive eigenvalue which exceeds every other eigenvalue in absolute value, and the corresponding space of eigenvectors is one-dimensional and is spanned by a vector with strictly positive coordinates. Numerous generalizations of these results to order-preserving linear operators acting in ordered linear spaces have appeared in recent years; a short bibliography is included at the end of this paper. In this paper a generalization in a different direction is obtained which reduces, in the finite dimensional case, to the assertion that the Perron-Frobenius theorems hold if it is only required that all but a finite number of the powers of the matrix satisfy the given conditions. The principal results are theorems of the Perron-Frobenius type which are applicable to any compact linear operator (the compactness condition is weakened somewhat), acting in an ordered real Banach space B, which satisfies a condition weaker than order-preserving. In addition, the results apply to the case when the "cone" of positive elements in B has no interior.

2. Preliminaries. Throughout the sequel, B will denote a real Banach space with norm $\|\cdot\|$. The complex extension of B, $\tilde{B}$, is the complex Banach space $\tilde{B} = \{x + iy \mid x, y \in B\}$ with the obvious definitions of addition and complex scalar multiplication and the norm in $\tilde{B}$ is $\| x + iy \| = \sup_\theta \| \cos\theta \cdot x + \sin\theta \cdot y \|$. If T is a (real) linear operator on B into B, the (complex) linear operator $\tilde{T}$ on $\tilde{B}$ into $\tilde{B}$ is defined by $\tilde{T}(x + iy) = Tx + iTy$. T is bounded if and only if $\tilde{T}$ is bounded, in which case $\| T \| = \| \tilde{T} \|$. The spectrum, $\sigma(T)$, and the resolvent, $\rho(T)$, are defined to be the corresponding sets associated with the operator $\tilde{T}$. We denote the spectral radius of T by r_T, $r_T = \lim_{n\to\infty} \| T^n \|^{1/n} = \sup_{\lambda \in \sigma(T)} |\lambda|$ (provided $\| T \| < \infty$).

In all of our results there will be a basic assumption that the linear operator under consideration is quasi-compact, a notion which we will now define. A bounded linear operator T is compact (also called completely continuous) if each sequence $Tx_1, Tx_2, \cdots,$ with

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