BASIC SEQUENCES AND THE PALEY-WIENER CRITERION

JAMES R. RETHERFORD

1. Introduction. Throughout the paper X will denote a complete metric linear space (i.e., a complete topological linear space with topology derived from a metric d with the property that $d(x, y) = d(x - y, 0)$, for all $x, y \in X$) or some specialization thereof over the real or complex field; $\|x\|$ will denote $d(x, 0)$; and if $\{x_n\}$ is a sequence in X, $[x_n]$ will denote the closed linear span of the elements $\{x_n\}_{n \in \omega}$.

A sequence $\{x_n\}$ is said to be a *basic sequence of vectors* if $\{x_n\}$ is a basis of vectors of the space $[x_n]$, i.e., for each $x \in [x_n]$ there corresponds a unique sequence of scalars $\{a_i\}$ such that

$$x = \sum_{i=1}^{\infty} a_i x_i \, , \tag{1.1}$$

the convergence being in the topology of X. We say that the basis is unconditional if the convergence in (1.1) is unconditional. It is well known that if $\{x_n\}$ is a basic sequence of vectors, then every $x \in [x_n]$ can be represented in the form $x = \sum_{i=1}^{\infty} f_i(x) x_i$ where $\{f_i\}$ is the sequence of continuous coefficient functionals biorthogonal to $\{x_i\}$ (Arsove [1, p. 368], Dunford and Schwartz [4, p. 71]).

Similarly, we say that a sequence $\{M_i\}$ of nontrivial subspaces of a complete metric linear space X is a *basis of subspaces* of X, if for each $x \in X$, there corresponds a unique sequence $\{x_i\}$, $x_i \in M_i$ for each i, such that

$$x = \sum_{i=1}^{\infty} x_i \, . \tag{1.2}$$

This concept has been studied by Fage [5], Markus [9], and others in separable Hilbert space and by Grimblyum [6] and McArthur [10] in complete metric linear spaces. We say that the basis of subspaces is *unconditional* if the convergence in (1.2) is unconditional.

If $\{M_i\}$ is a basis of subspaces for X, for each $i \in \omega$ define E_i from X into X by $E_i(x) = x_i$ where $\sum_{i=1}^{\infty} x_i$ is the unique representation of $x \in X$. E_i is a projection (linear and idempotent); $E_i E_j = 0$ if $i \neq j$; the range of E_i is M_i; for each $x \in X$, $x = \sum_{i=1}^{\infty} E_i(x)$ and if $E_i(x) = 0$ for each i, then $x = 0$. $\{M_i\}$ will be called a *Schauder basis of subspaces* if each E_i is continuous.

September 24, 1962. This research was supported by the Air Force Office of Scientific Research.