EXTREMAL ELEMENTS OF THE CONVEX CONE B_n OF FUNCTIONS

E. K. McLachlan

Let B_0 be the set of nonnegative real continuous on $[0, 1]$, let B_1 be the set of functions belonging to B_0 such that $\Delta_h^1 f(x) = f(x + h) - f(x) \geqq 0$, $h > 0$, for $[x, x + h] \subset [0, 1]$, and let B_n, $n > 1$ be the set of functions belonging to B_{n-1} such that $\Delta_h^n f(x) \geqq 0$ for $[x, x + nh] \subset [0, 1]$ [1]. Since the sum of two functions in B_n belongs to B_n and since a nonnegative real multiple of a B_n function is a B_n function, the set of B_n functions form a convex cone. It is the purpose of this paper to give the extremal elements [2] of this cone, to prove that they are not dense in a compact convex set that does not contain the origin but meets every ray of the cone, and to show that for the functions of the cone an integral representation in terms of extremal elements is possible. The intersection of the B_n cones is the well-known class of functions, the absolutely monotonic functions. Thus the set of these functions form a convex cone also. The extremal elements for this convex cone are given too.

In some correspondence with the author relative to the convex cone B_2, Professor F. F. Bonsall noted that the extremal elements of B_2 were the indefinite integrals of the characteristic functions that are extremal elements of the weak closure of B_1. Professor Bonsall guessed that successive integration would give the extremal elements of B_n. This proved to be a very good guess, and the author gratefully acknowledges the assistance of these comments.

In the following discussion the vertex of the convex cone is not considered as an extremal element.

1. The convex cone B_0. For $f \in B_0$, then take $f_1(x) = x f(x)$ and $f_2 = f - f_1$. Then f is the sum of functions in B_0 that are not proportional to f. Therefore, B_0 has no extremal elements.

2. The convex cone B_1. For $f = c > 0$ and $f = f_1 + f_2$ where f_1 and $f_2 \in B_1$ then $0 = \Delta_h^1 f(x) = \Delta_h^1 f_1(x) + \Delta_h^1 f_2(x)$ implies $\Delta_h^1 f_i(x) = 0$ for $i = 1, 2$ and $[x, x + h] \subset [0, 1]$. Therefore $f_i = c_i$, $c_i > 0$, $i = 1, 2$, where $c_1 + c_2 = c$. Hence f is an extremal element of B_1. Now $f = c > 0$ belongs also to B_n for $n > 1$. The set B_n is a subcone of B_1 and hence $f = c$ is again an extremal element of B_n.

If f is not constant then $f(0) = m$ and $f(1) = M$ and a non-proportional decomposition can be given by taking $f_1(x) = \min(f(x), (1/2)(M + m))$

Received October 23, 1963.