A REPRESENTATION THEORY FOR A CLASS OF PARTIALLY ORDERED RINGS

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The lattice ordered rings known as f-rings, introduced by Birkhoff and Pierce in [1], have been studied very intensively in the last few years. In particular Pierce has shown in [4] that the f-rings without nonzero nilpotents are precisely the (isomorphic images of) lattice ordered subdirect unions of totally ordered rings with integrity, and Johnson in [2] has gone on to prove that any Archimedean f-ring with no nonzero nilpotents can be represented as a lattice ordered ring of continuous extended realvalued functions on a locally compact Hausdorff space.

Since many commonly occurring examples of partially ordered rings are not lattice ordered it is natural to ask whether these two results can be generalised so as to be independent of the lattice structure. Such a generalisation is given here when multiplication is assumed commutative.

Theorem 1 characterises the subdirect unions of totally ordered commutative rings with integrity; Theorem 2 sharpens this result and Theorem 3 completes the programme by extending Johnson's representation.

The plan of the paper is as follows:

Section 1 is an introduction to the subject matter and methods of the paper; the succeeding three sections contain proofs of Theorems 1, 2 and 3 respectively and § 5 shows that for f-rings the representations given preserve the lattice structure.

1. Introduction. Throughout this paper "ring" will be an abbreviation for "commutative associative ring".

A *partially ordered* (or *po-*) *ring* is a ring whose elements are partially ordered in such a way that if $a \geqq b$ then $a + c \geqq b + c$ for all c and $ac \geqq bc$ for all $c \geqq 0$. Among the *po-rings* those with integrity (i.e. without divisors of zero) and a total ordering (the *toi-rings*) are particularly simple and it is our first aim to find out when a *po*-ring can suitably be built up from *toi*-rings. To make this more precise:

If $\{R_i\}_{i \in I}$ is a nonempty family of *toi*-rings their *direct union*, $\sum R_i$, is formed by taking the class of all functions $a: I \to \bigcup R_i$ with $a(i) \in R_i$ for all i, and defining addition by $(a + b)(i) = a(i) + b(i)$ for all i; multiplication by $(ab)(i) = a(i)b(i)$ for all i, and order by $a \geqq b$

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