MAXIMAL ALGEBRAS AND A THEOREM OF RADÓ

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1. A theorem of Radó [1, 4, 6, 9] asserts that a function f, continuous on the closed disc $D = \{z : |z| \leqq 1\}$, and analytic at all points of the interior of D where f doesn't vanish, is analytic on all the interior. One can of course take this as a statement about the uniformly closed algebra A_1—the disc algebra—formed by those f in $C(D)$ analytic on the interior of D, and in fact it is easy to restate the result in a form which makes sense for any function algebra. For let $T^1 = \{z : |z| = 1\}$, and call f locally approximable at z if f can be uniformly approximated by elements of A_1 on some neighborhood of z. Then it is clear that the result asserts that any f in $C(D)$, locally approximable at all z in $D \backslash (T^1 \cup f^{-1}(0))$, is in A_1.

Now since D can be viewed as the maximal ideal space of A_1, and T^1 as the Šilov boundary, we can formulate such an assertion for any uniformly closed algebra of functions—and, needless to say, it will fail in general.[1] But under appropriate maximality conditions the result does hold; in particular we shall show it holds for any uniformly closed function algebra A maximal on its Šilov boundary, provided the boundary is not all the maximal ideal space of A, and for intersections of such algebras.

This result holds as a consequence of two facts: Rossi's local maximum modulus principle [11], and a quite elementary lemma (2.1) which allows one to eliminate certain points as candidates for elements of the Šilov boundary of an algebra. In the original setting, where the elementary local maximum modulus principle for analytic functions can be used, our proof requires (beyond this lemma) only the fact that the disc algebra A_1 is a maximal subalgebra of $C(T^1)$ [7, 12]; no doubt it is no simpler than the proof given in [6]. However our arguments do establish some nontrivial variants of the result in the general setting (3.5, 3.6, 4.9), and, in particular, for functions analytic on polycylinders in C^n; deflated to the disc algebra almost all of these follow rather easily from Radó's result due to the topological simplicity of the one (complex) dimensional situation and the fact that there Radó's result can be applied locally.

One consequence of Radó's theorem is the fact that A_1 is *integrally closed* in $C(D)$, i.e., any f in $C(D)$ satisfying a polynomial equation

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[1] For example, for the subalgebra of A_1 of those f with $f'(0) = 0$; $f(z) \equiv z$ is locally approximable off $f^{-1}(0)$, but not in the subalgebra.