CLIFFORD VECTORS

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In this paper we present a generalization of parallel vector fields in a Riemannian space. As it turns out, such fields exist in spaces of constant positive curvature.

Restricting ourselves to a Riemannian 3-space throughout, we need the oriented third-order tensor [3, p. 249]

$$\eta_{ijh} = [\operatorname{sgn}(g)g]^{1/2}\varepsilon_{ijh} .$$

whose covariant derivative vanishes [3, pp. 251–252]. The latter fact is best ascertained by the use of geodesic coordinates. If we write the determinant of the metric tensor with the aid of permutation symbols we also find without difficulty

$$(1) \qquad g^{pq}\eta_{ijp}\eta_{khq} = g_{hj}g_{ik} - g_{hi}g_{jk} .$$

DEFINITION. *Let the direction of a vector field at any point be that of the unit vector* V. *The field is said to consist of Clifford vectors if*

$$(2) \qquad V_{i,j} = L^{\eta}{}_{ijh}V^h , \qquad L \neq 0 .$$

THEOREM. *If the Riemannian curvature* K *is constant and equal to* L^2, *the system of equations* (2) *is completely integrable. If, at any point, solutions of* (2) *exist in all directions, then* $K = L^2 = const.$

It is known that integrability conditions for (2) are obtained using covariant differentiation. Hence, on account of a Ricci identity [3, p. 83] and (1) we have

$$(3) \qquad L_{,k}\eta_{ijh}V^h - L_{,j}\eta_{ikh}V^h + L^2(g_{hj}g_{ik} - g_{hk}g_{ij})V^h = R_{hijk}V^h .$$

If the Riemannian curvature is constant [3, p. 112],

$$(4) \qquad R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij})$$

and conditions (3) are identically satisfied. This proves the first part of our theorem.

For proof of the second part we multiply (3) by $W^iV^jW^k$ and get

$$L^2(g_{hj}g_{ik} - g_{hk}g_{ij})V^hW^iV^jW^k = R_{hijk}V^hW^iV^jW^k .$$

Thus L^2 is the Riemannian curvature associated with the unit vectors

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