A NOTE ON REFLEXIVE MODULES

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For any ring A and left (resp. right) A-module E we let E^* denote the right (resp. left) A-module $\mathrm{Hom}_A(E, A_s)$ (resp. $\mathrm{Hom}_A(E, A_d)$) where A_s (resp. A_d) denotes A considered as a left (resp. right) A-module. Then the mapping $E \rightarrow E^{**}$ such that $x \in E$ is mapped onto the mapping $\varphi \rightarrow \varphi(x)$ is linear.

Specker [3] has shown that if E is a free Z-module with a denumerable base (where Z denotes the ring of integers) then E is reflexive, i.e. the canonical homomorphism $E \rightarrow E^{**}$ is a bijection. In this paper it is shown that a free module E with a denumerable base over a discrete valuation ring A is reflexive if and only if A is not complete and if and only if E is complete when given the topology having finite intersections of the kernels of the linear forms as a fundamental system of neighborhoods of 0. Specker's result can be deduced from these results. We note that this topology has been used and studied by Nunke [2] and Chase [1].

THEOREM 1. *Let A be a discrete valuation ring with prime Π and let E be a free A-module with a denumerable base. Then E is reflexive if and only if A is not complete.*

Proof. Let $(a_i)_{i \in N}$ (N the set of natural numbers) be a base of E and let $E_j = \{\varphi \mid \varphi \in E^*,\ \varphi(a_i) = 0,\ i = 0, 1, 2, \cdots, j-1$. Let $a'_j \in E^*$ be such that $a'_j(a_j) = 1$, $a'_j(a_k) = 0$ if $j \neq k$. Then clearly a'_0, $a'_1, \cdots a'_{j-1}$ generate a supplement of E_j in E^*. For each $x \in E$ the canonical image of x in E^{**} annihilates some E_j and conversely if $\psi \in E^{**}$ annihilates E_j then ψ is the canonical image of $\sum_{i=0,1,\cdots,j-1} \psi(a'_i)ai$. Hence $E \rightarrow E^{**}$ is a surjection if and only if each $\psi \in E^{**}$ annihilates some E_j. If $E \rightarrow E^{**}$ is not a surjection let $\psi \in E^{**}$ be such that $\psi(E_j) \neq 0$ for each $j \in N$ and let $\varphi_j \in E_j$ be such that $\psi(\varphi_j) \neq 0$. We can suppose that $\varphi_j \in \Pi^j E_j$ and that $\psi(\varphi_j) \in \Pi^{m}_{j} A$ but $\psi(\varphi_j) \notin \Pi^{m_j+1} A$ where $m_{i+1} > m_i$ for all $i \in N$. To show A complete it suffices to show that every series $\sum_{j \in N} \beta_j \Pi^{mj}$, $\beta_j \in A$ converges. We can find a scalar multiple of φ_j say φ'_j such that $\psi(\varphi'_j) = \beta_j \Pi^{m}_{j}$. Then let $\varphi \in E^*$ be such that $\varphi(x) = \sum_{j \in N} \varphi'_j(x)$ for all $x \in E$. This sum is defined since for a fixed $x \in E$ and M sufficiently large positive integer we have $\varphi_{M+i}(x) = 0$ for all $i \in N$. Furthermore, since $\varphi'_j \in \Pi^j E_j$ it is clear that the series $\sum \varphi'_j$ converges to φ when E^* is given the topology having

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