ON CONTINUOUS MATRIX-VALUED FUNCTIONS ON A STONIAN SPACE

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1. Introduction. In this paper the authors continue the study (begun in [9] and carried on in [3] and [10]) of matrices with entries from the algebra $C(\mathfrak{X})$ of all continuous complex-valued functions on an extremely disconnected, compact Hausdorff space $\mathfrak{X}$. (Such spaces are sometimes called Stonian after M. H. Stone, who considered them in [14].) One of the authors has shown ([10], Theorem 3) that if A and B are $n \times n$ matrices over $C(\mathfrak{X})$ such that $A(x)$ is unitarily equivalent to $B(x)$ for each $x \in \mathfrak{X}$, then A and B are unitarily equivalent in the algebra $M_n(\mathfrak{X})$ of all $n \times n$ matrices over $C(\mathfrak{X})$. It is thus natural to ask whether the similarity of $A(x)$ and $B(x)$ for each $x \in \mathfrak{X}$ is sufficient to guarantee the similarity of A and B in $M_n(\mathfrak{X})$. We show by example in § 2 that the answer is no; however, we also show that if the hypothesis is strengthened by the addition of a uniform boundedness requirement, then the similarity of A and B in $M_n(\mathfrak{X})$ does indeed follow. As a by-product of the technique introduced to give this result, we obtain a new short proof of Theorem 3 of [10].

In § 3 we show that a certain class of entire functions maps $M_n(\mathfrak{X})$ onto itself; this is a generalization (with a different proof) of a result of Kurepa [8] for $n \times n$ matrices, and adds to the information obtained by Brown [1] on the question of which entire functions map which Banach algebras onto themselves. As a corollary, we learn that every invertible element of $M_n(\mathfrak{X})$ has a logarithm. Section 4 is devoted to proving that an element of $M_n(\mathfrak{X})$ has an identically vanishing trace if and only if it is a commutator in $M_n(\mathfrak{X})$. (See Remark 2, § 4, for a paraphrase of this result cast in the terminology of operator theory on Hilbert space.) Finally, in § 5 the authors give two examples which indicate that it is probably fruitless to pursue the structure theory of matrices over $C(\mathfrak{X})$ where $\mathfrak{X}$ is a more general topological space than a Stonian space.

2. Similarity in $M_n(\mathfrak{X})$. The most convenient definition of $M_n(\mathfrak{X})$ is as follows. Let M_n denote the full ring of $n \times n$ complex matrices under the operator norm, and let $\mathfrak{X}$ be any Stonian space. Denote by $M_n(\mathfrak{X})$ the *-algebra of continuous functions from $\mathfrak{X}$ to M_n, where the algebraic operations in $M_n(\mathfrak{X})$ are defined pointwise. Under the norm $\|A\| = \sup_{x \in \mathfrak{X}} \|A(x)\|$, $M_n(\mathfrak{X})$ is a C^*-algebra identifiable with the C^*-algebra of all $n \times n$ matrices over $C(\mathfrak{X})$. Moreover, $M_n(\mathfrak{X})$ is an

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