INTEGRAL INEQUALITIES FOR FUNCTIONS WITH NONDECREASING INCREMENTS

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1. Introduction. One of the fundamental inequalities of analysis is Jensen's inequality,

$$\int f(x)\,dG(x) \geqq f\left(\int x\,dG(x)\right), \tag{1.1}$$

for convex f, with G a probability distribution function. However, G need not be a probability distribution function in order that (1.1) hold for all convex f. Let $X(t)$ be nondecreasing for $\alpha \leqq t \leqq \beta$. It was shown in [1] that under mild regularity conditions on G, if $G(\alpha) = 0$, necessary and sufficient conditions for

$$\int_\alpha^\beta f[X(t)]\,dG(t) \geqq f\left(\int_\alpha^\beta X(t)\,dG(t)\right) \tag{1.2}$$

for all convex f are

$$G(\beta) = 1\,, \tag{1.3}$$

and

$$\int_\alpha^t G(u)\,dX(u) \geqq 0\,, \quad \int_t^\beta [1 - G(u)]\,dX(u) \geqq 0 \quad \text{for } \alpha \leqq t \leqq \beta\,. \tag{1.4}$$

This result was applied to show that:

(i) sufficient conditions in order that (1.2) hold for convex f are $X(\alpha) = 0, f(0) \leqq 0$, and $0 \leqq G(t) \leqq 1$ for $\alpha \leqq t \leqq \beta$; and

(ii) if f is convex on $[0, b]$ with $f(0) \leqq 0$, if $0 \leqq a_1 \leqq \cdots \leqq a_m \leqq b$, if $0 \leqq h_1 \leqq \cdots \leqq h_m \leqq 1$, then

$$\sum_{j=1}^m (-1)^{j-1} h_j f(a_j) \geqq f\left[\sum_{j=1}^m (-1)^{j-1} h_j a_j\right]. \tag{1.5}$$

The latter, (ii), was proved independently by Olkin [5]. Ciesielski [2] obtained results (under unnecessarily stringent hypotheses) related to (i) through change of variable, and obtained also analogous two-dimensional results. These provided part of the motivation for the present study of k-dimensional analogues of (1.2).

In the present paper, $X(\cdot)$ denotes a map from the real interval $[\alpha, \beta)$ into an interval I in k-dimensional Euclidean space R^k such that each component of X is nondecreasing. The function f is a map from

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