BOUNDS FOR DERIVATIVES IN ELLIPTIC BOUNDARY VALUE PROBLEMS

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I. Introduction. In a recent paper [7], Payne and Weinberger gave pointwise bounds for solutions of second order uniformly elliptic partial differential equations. The bounds for the function and its gradiant involved derivatives of the boundary data. Later [2] the present authors gave a method for obtaining bounds in which no derivatives of the boundary data appeared. Pointwise bounds for derivatives were not dealt with. In [4] the authors gave a method for bounding derivatives for Poisson's equation. The method was, however, restricted to the Laplace operator (or the constant coefficient case) and was not generally applicable.

In this paper we consider the operator

$$Lu \equiv (a^{ij}u_{,i})_{,j} \tag{1.1}$$

where u is a sufficiently smooth function defined in some region R (with boundary C) of Euclidean N dimensional space. Here the notation $u_{,i}$ denotes the partial derivative of u with respect to the cartesian coordinate x^i. In (1.1) the summation convention is used, i.e. $(a^{ij}u_{,i})_{,j} \equiv \sum_{i\ j=1}^{N} (a^{ij}u_{,i})_{,j}$. The coefficient matrix a^{ij} may be a function of position and is assumed to be uniformly positive definite and bounded above. That is there exist positive constant a_0 and a_1 such that

$$a_0 \sum_{i=1}^{N} \xi_i^2 \leqq a^{ij}\xi_i\xi_j \leqq a_1 \sum_{i=1}^{N} \xi_i^2 \tag{1.2}$$

for any real vector $\xi = (\xi_1, \cdots, \xi_N)$. We shall give a method involving the use of a parametrix, for obtaining bounds on any derivative of a function u at an arbitrary interior point P of R. These bounds are in terms of Lu and $\max_{S(P)} |u|$, where $S(P)$ is a sphere containing P. Estimates of this type for very general elliptic operators are described by John [6]. His method does not involve the parametrix and hence the expressions which could be derived would turn out to be quite different. Thus the problem is reduced to that of bounding $\max_{S(P)} |u|$ in terms of quantities which are data of some boundary value problem. We assume throughout that Lu and the coefficients a^{ij} are sufficiently smooth so that all subsequent indicated operations are valid.

In this paper we concern ourselves only with the derivation of appropriate a priori inequalities. The manner of applying such ine-

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