FAITHFUL *-REPRESENTATIONS OF NORMED ALGEBRAS II

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1. Introduction. Let A be a complex Banach algebra with an involution $x \to x^*$. By the positive cone P of A is meant the closure, in the set H of self-adjoint elements of A, of the set of all finite sums of elements of the form x^*x. Kelley and Vaught [5] have shown that, if A has an identity,[1] A has a faithful *-representation (as bounded linear operators on a Hilbert space) if and only if (1) $x \to x^*$ is continuous and (2) $P \cap (-P) = (0)$. Consider the (incomplete) normed algebra case. Examples exist with a faithful*-representation and both conditions false, with (1) true and (2) false, and with (1) false and (2) true. Moreover, even if (1) holds so that $x \to x^*$ extends to the completion A_c of A, one can have a continuous faithful *-representation for A when none exists for A_c. It follows that the results which we now describe, even for the normed algebra case, can *not* be deduced from the theory of Banach algebras.

These facts led us to consider the development of a theory of *-representations of a complex algebra A with involution (with or without an identity) under minimal assumptions on A but with results sufficiently definitive to illuminate the counter-examples mentioned above. We suppose that the real linear space H has a norm in terms of which it is a real normed linear space such that

(a) the real subalgebra generated by each $h \in H$ is a normed algebra and

(b) the Jordan product $x \cdot h = xh + hx$ is a continuous function on H for each fixed $h \in H$.

It is shown that A has a faithful *-representation continuous on H if and only if A is semi-simple and $P \cap (-P) = (0)$. If A is a normed *-Q-algebra, any *-representation is automatically continuous on H so that these conditions are necessary and sufficient there for a faithful *-representation. As already noted, this can fail if the Q-algebra hypothesis is dropped.

For previous work on *-representations we refer to [5], [7], [8], and [10].

2. Preliminaries. Let A be an algebra over the complex field

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[1] As pointed out in [**10**, p. 352] this statement is incorrect if A has no identity. For a version covering that case see [**10**, Theorem 3.4]. Theorem 4.3 below shows that A has a faithful *-representation if and only if A is semi-simple and $P \cap (-P) = (0)$.