BOUNDED GENERALIZED ANALYTIC FUNCTIONS ON THE TORUS

VICTOR L. SHAPIRO

1. Introduction. We shall operate in Euclidean k-space, E_k, $k \geqq 2$, and use the following notation:

$$x = (x_1, \cdots, x_k) \; ; \qquad y = (y_1, \cdots, y_k) \; ;$$
$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \cdots, \alpha x_k + \beta y_k) \; ;$$
$$(x, y) = x_1 y_1 + \cdots + x_k y_k \; ; \qquad |x| = (x, x)^{1/2} .$$

T_k will designate the k-dimensional torus $\{x; -\pi < x_j \leqq \pi, j = 1, \cdots, k\}$, v will always designate a point a distance one from the origin, i.e., $|v| = 1$, and m will always designate an integral lattice point. If f is in L^1 on T_k, then $\hat{f}(m)$ will designate the mth Fourier coefficient of f, i.e., $(2\pi)^{-k} \int_{T_k} f(x) e^{-i(m,x)} dx$.

We shall say that $f(x)$ in L^1 on T_k is a generalized analytic function on T_k if there exists v such that f is in A_v, where $A_v = A_v^+ \cup A_{-v}^+$, and A_v^+ is defined as follows:

f is in A_v^+ if there exists an m_0 such that if $m \neq m_0$ and $(m - m_0, v) \leqq 0$, then $\hat{f}(m) = 0$.

We shall say that $f(x)$ in L^1 on T_k is a strictly generalized analytic function on T_k if there exists a v such that f is in B_v, where $B_v = B_v^+ \cup B_{-v}^+$, and B_v^+ is defined as follows:

f is in B_v^+ if there exists an m_0 and a γ with $0 < \gamma < 1$ such that if $(m - m_0, v) < \gamma |m - m_0|$, then $\hat{f}(m) = 0$.

It is quite clear that $B_v \subset A_v$. In this paper, we shall obtain a result which is false for bounded functions in A_v but which is true for bounded functions in B_v. It is primarily with the class B_v and its extension to finite complex measures that the classical paper of Bochner [2, p. 718] is concerned. On T_k, it is essentially with the class A_v that the papers of Helson and Lowdenslager [5], [6], and de Leeuw and Glicksberg [4] are concerned.

We shall be concerned in this paper with a class of functions C_v which for bounded functions is intermediate between the two classes B_v and A_v.

We first note that if f is in B_v^+, then $\sum_m |\hat{f}(m)| e^{(m,v)\sigma} < \infty$ for every $\sigma < 0$. For with $||f||_p$, $1 \leqq p \leqq \infty$, designating the L^p-norm of f on T_k, we see that there exists a γ with $0 < \gamma < 1$ and an m_0 such that

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