ON CONTINUITY OF MULTIPLICATION IN A COMPLEMENTED ALGEBRA

PARFENY P. SAWOROTNOW

The present study was originally motivated by reading a paper of M. Rajagopalan [6]. The author and Sr. K. A. Bellcourt were able to obtain the same result as M. Rajagopalan under much weaker hypothesis. Latter the author realized that in the case of an H^*-algebra and a two-sided H^*-algebra the condition "$\|xy\| \leqq M\|x\|\cdot\|y\|$" is a consequence of the other axioms in the definition. The same is true about right H^*-algebra if we assume continuity of involution (It was pointed out to the author that P. J. Laufer established this result in 1958. The author arrived at it independently of Laufer).

The present paper deals with the question whether the same is true about complemented algebras. It turns out that we have to assume topological semi-simplicity in some sense and continuity of the mapping $x \to xa$ (Sr. Bellcourt should be credited with the idea of assuming topological semi-simplicity). Below we have a new characterization of complemented algebras. Lemma 1 may be of interest by itself.

2. LEMMA 1. *Let A be an* (associative) *algebra whose underlying vector space is a Banach space* (in other words A would be a Banach algebra if we would assume that $\|xy\| \leqq M\|x\|\cdot\|y\|$). *Suppose that the mapping R_a: $x \to xa$ is continuous for each $a \in A$; suppose also that the mapping L_a: $x \to ax$ is continuous for each a in some dense subset B of A. Then A is a Banach algebra.*

Proof. Let $a \in A$ and let a_n be a sequence of members of B such that $a_n \to a$. Then the sequence $\|a_n\|$ is bounded; also the sequence $\|a_nx\|$ is bounded for each x in A ($\|a_nx\| \leqq \|R_x\|\cdot\|a_n\|$). From Theorem 5, page 80, of [3] we may conclude that there exists a positive number M such that $\|L_{a_n}\| \leqq M$ for each n.

Now let $x_m, x \in A$ be such that $x_m \to x$. Then $\|ax - a_nx_m\| = \|ax - a_nx + a_nx - a_nx_m\| \leqq \|a - a_n\|\cdot\|R_x\| + M\|x - x_m\|$.

From this we may draw two conclusions. First of all $B = A$ (note that the above inequality implies that $ax_m \to ax$: $\|ax - ax_m\| \leqq \|ax - a_nx_m\| + \|a_n - a\|\cdot\|R_{x_m}\|$). Secondly, combining this fact with the above we see that the mapping $\langle x, y\rangle \to xy$ is continuous. This conclusion can be obtained also using a result of [2].

Received August 27, 1963.