NETS WITH CRITICAL DEFICIENCY

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This paper is a sequel to a paper of Bruck [2]. With only minor variations, we shall use Bruck's notation and terminology. All references to Bruck should be understood as references to [2].

For a net N of order n and deficiency d, the deficiency is said to be critical if $n = (d - 1)^2$. If d is less than the critical deficiency, Bruck shows that

(1) N can be extended to an affine plane of order n in at most one way.

(2) The number of distinct transversals is less than or equal to dn.

(3) N can be embedded in an affine plane of order n if equality holds in (2).

In this paper, we show that if the deficiency is critical, then

(1′) N can be extended to an affine plane of order n in at most two ways. If two planes are obtained, they are related to each other by a construction due to the author.

(2′) The number of distinct transversals is less than or equal to $2dn$.

(3′) N can be extended to a plane in two different ways if equality holds in (2′).

We also show that N can be extended to a plane in at most one way if the critical value is exceeded only slightly.

We are concerned with the possibility that N may be extended to an affine plane in more than one way. Suppose that N can be extended to a plane π by adjoining the lines of a complementary net N_1. Then, if T is a transversal which is not a line of N_1, we shall say that T is an *extra transversal* (with respect to N_1).

THEOREM 1. *If T is an extra transversal with respect to N_1 and $n > (d - 1)^2 - \frac{1}{2}(d - 1)$ then*

(1) $n = (d - 1)^2$

(2) *T is a subplane of N_1 of order $d - 1$.*

Proof. Let p be a point belonging to T. We assert that every line of N_1 which goes through p must contain more than $\frac{1}{2}(d - 1)$ points of T: The lines of N_2 are transversals (of N); by Bruck's Lemma 3.2, no line of N_1 can contain more than $d - 1$ points of T. In the extreme case, suppose that $d - 1$ lines of N_1 through p were to each

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