A GENERALIZATION OF POWER-ASSOCIATIVITY

J. MARSHALL OSBORN

Probably the most promising new identity to arise in a recent study of identities on commutative algebras [3] is

$$2((x^2\cdot x)x)x + (x^2\cdot x)x^2 = 3(x^2\cdot x^2)x \, . \tag{2}$$

This identity generalizes not only the power-associative identity, $x^2\cdot x^2 = (x^2\cdot x)x$, but also the generalization of the Jordan identity considered in [4]. In the present paper, we study the structure of commutative rings of characteristic relatively prime to 2, 3, 5, or 7 satisfying (1). This restriction on the characteristic will be assumed throughout the paper without further mention.

There are two obvious ways in which the structure theory of the class of rings studied here is noticeably weaker than the structure theory of power-associative rings. First of all, given a ring A satisfying (1) containing an idempotent e, there can exist elements of A which are annihilated by the operator $(2R_e - I)^2$ but not by $(2R_e - I)$. Secondly, defining the additive subgroups $A_\lambda = A_e(\lambda) = \{x \mid x \in A, \; xe = \lambda x\}$ for $\lambda = 0$, $1/2$, and 1, the relations $A_1A_0 = 0$ and $A_{1/2}A_{1/2} \subset A_1 + A_0$ are not valid in general. Despite these impediments, we see in §1 that A may be decomposed simultaneously with respect to a set of mutually orthogonal idempotents in much the usual fashion. In §2 we prove that, if A is simple of degree ≥ 3 satisfying the condition that $x(2R_e - I)^2 = 0$ if and only if $x(2R_e - I) = 0$ for all x in A, then A is a Jordan ring.

1. We begin our investigation by partially linearizing (1) to obtain

$$\begin{aligned} 4((yx\cdot x)x)x + 2(yx^2\cdot x)x + 2yx^3\cdot x + 2y(x^3\cdot x) + 2(yx\cdot x)x^2 \\ + yx^2\cdot x^2 + 2yx\cdot x^3 = 12(yx\cdot x^2)x + 3y(x^2\cdot x^2) \, . \end{aligned} \tag{2}$$

Then, setting $x = e$ in (2) immediately yields

$$4yR_e^4 - 8yR_e^3 + 5yR_e^3 - yR_e = 0, \quad \text{or}$$

$$y[(R_e - I)(2R_e - I)^2R_e] = 0 \, . \tag{3}$$

Defining $B_{1/2} = B_e(1/2) = \{x \mid x \in A, \;\; x(2R_e - I)^2 = 0\}$, it follows from (3) that A may be decomposed into the additive direct sum

$$A = A_1 + B_{1/2} + A_0 \, . \tag{4}$$

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