KLEENE QUOTIENT THEOREMS

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0. Introduction. The collection of *regular subsets* of a semigroup A is the smallest collection of subsets of A having among its members the finite subsets of A, the collectionwise products EF and unions $E \cup F$ of any members E and F of it, and the subsemigroups E^* of A generated by each of its members E. For convenience, we set $\varnothing^* = \varnothing$ for the empty set $\varnothing$ although a semigroup is required to have at least one element. By a *quotient* of a semigroup A we shall mean, as usual, the set of inverse images of elements of a homomorphic image of A with multiplication so defined that this partition of A is isomorphic to the homomorphic image in the natural way.

A theorem of S. C. Kleene [3], first proved a dozen years ago, not only may be regarded as the fundamental theorem of the traditional theory of finite automata (compare C. C. Elgot [2] and Section 4 below), but it may be considered a contribution to the theory of quotients of certain classes of semigroups. Kleene's results are often summarized [2] by a statement equivalent to the following: *The regular subsets of a finitely generated free semigroup are the unions of subsets of finite quotients of the semigroup.* This result has two evident parts. The first we call the *Kleene Quotient Theorem*: *Each element of a finite quotient of a finitely generated free semigroup is a regular subset of the semigroup.* The second is one of several theorems we call the *Converse Theorems*: *Every regular subset of a finitely generated free semigroup is a union of elements of some finite quotient of the semigroup.* This result may be decomposed into two main parts (Section 3), each of which is also a Converse Theorem:

(a) *The collection wise product of unions of elements of finite quotients of a finitely generated free semigroup is a union of elements of a finite quotient of the semigroup.*

(b) *The subsemigroup generated by a union of elements of a finite quotient of a finitely generated free semigroup is a union of elements of a finite quotient of the semigroup.*

Our main purposes have been to remove the word *free* from the Kleene Quotient Theorem and to remove the adjective *finitely generated* from all the Converse Theorems. We have indeed been able to give some general inductive formulas (Section 1) which may themselves be regarded as a generalization of the Kleene Quotient Theorem, and we

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