ON THE NORMAL BUNDLE OF A MANIFOLD

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In the Michigan lecture notes of 1940 [8] Whitney proved that any manifold in the cobordism class of P_2 cannot be embedded in R^4 with a normal field while non-orientable manifolds in the trivial cobordism class may or may not have a normal field. We will give a new proof of this result using some of the recent notions of differential topology. As one would expect, Whitney's theorem is a special case of a more general theorem and for the statement of this theorem we introduce some notation.

Let M^n be a compact smooth n-manifold. Let $\bar{w}_i$ be the dual Stiefel Whitney classes of M^n.

DEFINITION. Let $\sigma(M^n) = 0$ if $\bar{w}_1 \cdot \bar{w}_{n-1} = 0$ and $\sigma(M^n) = 1$ if $\bar{w}_1 \cdot \bar{w}_{n-1} \neq 0$.

Clearly $\sigma(M^n)$ is just a Stiefel Whitney number [6]. Note also that by a result of Massey [5], $\sigma(M^n) = 0$ unless $n = 2^j$.

THEOREM 1. *For any embedding of M^n in R^{2n} the (twisted) Euler class is congruent to 2σ* mod 4.

This result is a slight sharpening of the theorem of Massey [4]; the proof is given in § 4 after some preliminary results in §§ 2 and 3.

Let χ be the Euler characteristic of M^2. In Whitney's theorem the role of σ in Theorem 1 is played by χ. It is not hard to verify that for 2-dimension manifolds $\sigma = \chi \bmod 2$. In addition, for 2-dimensional manifolds we can prove (section 6)

THEOREM 2. *For each k and each value of σ there is a manifold M^2 and an embedding of M^2 in R^4 with twisted Euler class $2\sigma + 4k$.*

We have not been able to show that a single manifold has an embedding for each k. Whitney exhibited two embeddings of the Klein bottle, one with a trivial Euler class and one with a non-trivial one.

We also have this weaker result (section 7) for other values of n.

THEOREM 3. *For every even n there exists a manifold M^n and an embedding of M^n in R^{2n} with no normal field.*

It is known that if $n \neq 2^j$ and $n > 3$, then every n-manifold embeds

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