A CHARACTERIZATION OF EXTREMALS FOR GENERAL MULTIPLE INTEGRAL PROBLEMS

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1. Introduction. The theory of quadratic functionals on a Hilbert space is applied here to variational problems for multiple integrals, involving several dependent variables and their partial derivatives up to any finite order, in nonparametric form. The treatment is confined to integration over open sets with fixed boundary, and to weak neighborhoods of an extremal. The basic result is the establishment of general sufficiency theorems for a weak relative minimum with isoperimetric or differential side conditions. A specific application is made of the sufficiency theorem in the case of isoperimetric side conditions to the extension of a well known characterization of of extremals. This characterization has been treated by Poincaré [16], Birkhoff and Hestenes [1], Karush [9] and others. It is analogous to the characterization of saddle points on two-dimensional surfaces as constrained extrema.

2. Quadratic forms. The theory of quadratic functionals (forms) has been developed explicitly by Hestenes [6, 7], and implicitly by Van Hove [19] and writers on elliptic partial differential equations.

If $Q(x)$ is a quadratic form on a Hilbert space $\mathfrak{H}$ with real scalars then there exist unique subspaces $\mathfrak{H}_+, \mathfrak{H}_0, \mathfrak{H}_-$ of $\mathfrak{H}$, having the null vector as their only common element, that are mutually orthogonal and Q-orthogonal, are such that Q is positive on $\mathfrak{H}_+$, negative on $\mathfrak{H}_-$, and zero on $\mathfrak{H}_0$, and are such that $\mathfrak{H} = \mathfrak{H}_+ + \mathfrak{H}_0 + \mathfrak{H}_-$ [6, p. 543]. The sum of the dimensions of the subspaces $\mathfrak{H}_-$ and $\mathfrak{H}_0$ will be called the *isoperimetric index* of Q on $\mathfrak{H}$. A quadratic form that is representable on $\mathfrak{H}$ as the sum of a positive definite quadratic form and a w-continuous quadratic form has been called a Legendre form by M. R. Hestenes. The fact that the isoperimetric index of a Legendre form is finite is significant for the characterization of extremals given below.

If $Q(x)$ and $K(x)$ are quadratic forms on $\mathfrak{H}$ such that $J(b; x) = Q(x) + bK(x)$ is a Legendre form for every positive number b, $K(x) \leqq 0$, and $Q(x) > 0$ whenever $K(x) = 0$ and $x \neq 0$, then there is a positive number c such that $J(c; x)$ is positive definite on $\mathfrak{H}$. A corollary to this is: *If $Q(x)$ and $K(x)$ are quadratic forms on $\mathfrak{H}$, $J(b_0; x) = Q(x) + b_0K(x)$ is a Legendre form for some number b_0, $K(x) \geqq 0$, and*

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