NATURAL SUMS AND ABELIANIZING

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Introduction. It is a well known result, essentially due to Mac Lane [13], that the addition in an abelian category is determined by the multiplication, i.e. every categorical isomorphism is additive. The sum $f + g$ of two mappings $f: A \to B$, $g: A \to B$ is defined—in fact, overdefined—by the following diagram involving natural isomorphisms of free sums $A \vee A$ and direct products $A \times A$.

$$\begin{array}{ccccccc} & & A \vee A & \xrightarrow{f \vee g} & B \vee B & \xrightarrow{\nabla} & B \\ & & \updownarrow & & \updownarrow & & \\ A & \xrightarrow{\Delta} & A \times A & \xrightarrow{f \times g} & B \times B & & \end{array}$$

In many non-abelian categories, especially in the category of all groups, one has not isomorphisms but naturally distinguished mappings $A \vee A \to A \times A$. The definition of a sum $f + g$ becomes a problem, which for groups may be best posed in the form

$$\begin{array}{ccccccc} & & A \vee A & \xrightarrow{f \vee g} & B \vee B & \xrightarrow{\nabla} & B \\ & & \downarrow & & & \nearrow & \\ A & \xrightarrow{\Delta} & A \times A & \cdots & \cdots & & \end{array}$$

This form is chosen [6] because $A \vee A \to A \times A$ is an epimorphism $\twoheadrightarrow$, so that there is at most one mapping $\cdots\cdots>$ making the diagram commutative; f and g are called *summable* if this mapping exists, and then their sum is the composed mapping $A \to A \times A \to B$. This partially defined addition, called the *natural sum*, turns out to be the same as Fitting's [2] pointwise multiplication of homomorphisms with commuting values; it has been used in extensive investigations of direct decompositions by Kuroš [9] and Livšic [10, 11, 12].

Of course a dual diagram applies in categories which have naturally distinguished monomorphisms $A \vee A \rightarrowtail A \times A$. In many categories there is a naturally distinguished mapping $A \vee A \to A \times A$ which is neither epimorphic nor monomorphic. This is the situation in the category of homotopy types of topological spaces with base point, and the theory of homotopy operations, despite some analogies, apparently requires the apparatus currently in use there [1] which is not related to Mac Lane's fundamental diagram. However, Mac Lane's approach

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