A PROOF OF THE NAKAOKA-TODA FORMULA

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If X_j $(1 \leqq j \leqq r)$ are objects we denote the corresponding r-tuple $(X_1, X_2, \cdots, X_r)$ by X and the $(r-1)$-tuple $(X_1, X_2, \cdots, X_{i-1}, X_{i+1}, \cdots, X_r)$ by $X(i)$. When X_j $(1 \leqq j \leqq r)$ are based topological spaces ΠX will denote their topological product and $\Pi^i X$ the subspace of ΠX whose points have at least i coordinates at base points (always denote by $*$).

Let $\alpha_j \in \pi_{n_j}(X_j)$ $(n_j \geqq 2, 1 \leqq j \leqq r, r \geqq 3)$ be elements of homotopy groups then we have

$$\star\alpha(\text{say}) = \alpha_1\star\alpha_2\star \cdots \star\alpha_r \in \pi_n(\Pi X, \Pi^1 X) ,$$

where $n = \Sigma n_j$ and $\star$ denotes the product of Blakers and Massey [1]. We thus also have

$$\star\alpha(i) \in \pi_{n-n_i}(\Pi X(i), \Pi^1 X(i)) .$$

There is a natural map $\Pi X(i), \Pi^1 X(i) \to \Pi^1 X, \Pi^2 X$ and we denote also by $\star\alpha(j)$ its image induced in $\pi_{n-n_i}(\Pi^1 X, \Pi^2 X)$. Let ∂ denote the homotopy boundary homomorphism in the exact sequence of the triple $(\Pi X, \Pi^1 X, \Pi^2 X)$. We shall prove the formula:

$$\partial\star\alpha = \Sigma(1 \leqq i \leqq r)(-1)^{\varepsilon(i)}[\alpha_i, \star\alpha(i)] \in \pi_{n-1}(\Pi^1 X, \Pi^2 X) , \qquad (0.1)$$

where $\varepsilon(1) = 0$, $\varepsilon(i) = n_i(n_1 + n_2 + \cdots + n_{i-1})$ $(i > 1)$ and where the brackets refer to the generalised Whitehead product of Blakers and Massey [1]. In the case of the universal example 0.1 becomes the formula of Nakaoka and Toda stated in [4] and proved there for $r = 3$. I. M. James[1] has raised the question of its validity for $r > 3$ and as the formula has applications (see [2], [3]) it would seem desirable to have a proof available in the literature. The present argument while inspired by [4] has a few novel features.

(1) DEFINITIONS AND LEMMAS. Let $x = (x_1, x_2, \cdots, x_n)$ denote a point of n-dimensional Euclidean space and let

$$\begin{aligned} V^n &= \{x; \Sigma x_i^2 \leqq 1\} , \\ S^{n-1} &= \{x; \Sigma x_i^2 = 1\} , \\ E_+^{n-1} &= \{x \in S^n; x_n \geqq 0\} , \\ E_-^{n-1} &= \{x \in S^n; x_n \leqq 0\} , \\ D_+^n &= \{x \in V^n; x_n \geqq 0\} , \end{aligned}$$

Received November 12, 1963.

[1] Math. Reviews 25 # 4521.