# ON LOCAL PROPERTIES AND $G_\delta$ SETS

EDWARD E. GRACE

**1. Introduction.** For complete metric spaces, R. H. Bing has shown [2, Theorem 2] that being not connected im kleinen at each point of a dense $G_\delta$ set implies the existence of an open set that contains uncountably many components. The proof, in fact, shows that there is an open set no component of which contains an open set. For Baire topological continua, the author has shown [4, Lemma 2] that being not connected im kleinen, at each point of a dense-domain intersection set, implies the existence of an open set no component of which contains an open set.

For a certain class of $II_\phi$ spaces [3, p. 642], including complete metric spaces and Baire topological spaces, there is a general theorem about local properties which has both of these theorems as corollaries. For simplicity, the theory is presented here only as it applies to the special case of complete metric spaces. Generalization to $II_\phi$ spaces, which have "developments" [1, p. 180] consisting of $\phi$ (perhaps not $\aleph_0$) collections of open sets, is rather straightforward. The pattern of this generalization is indicated to some extent by [4].

This theory applies to those local properties which the space has at a point $x$ if and only if $x$ is a distinguished point in the following sense. There is a relation "is a *distinguished subset* of" having the following two properties. (1) $D'$ is a distinguished subset of $D$ only if $D'$ is open and is a subset of $D$. (2) If $D$ contains $D'$ and $D''$ is a distinguished subset of $D'$, then $D''$ is a distinguished subset of $D$. A point $x$ is said to be a *distinguished point* if each open set containing $x$ contains a distinguished subset containing $x$.

Several of the corollaries given here have been known for some time.

**2. Theorems.** The proof of Theorem 1 is complicated slightly in order that it will be clearer how it generalizes to apply to the general class of spaces referred to above.

THEOREM 1. *If $M$ is a complete metric space each open set of which contains a distinguished open subset, then the set of distinguished points of $M$ is a dense $G_\delta$ set.*

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