ON A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN FOUR VARIABLES

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1. Introduction. Bergman [1]-[6], [11] has considered the elliptic partial differential equation,

$$T_3[\Psi] = \frac{\partial^2 \Psi}{\partial x_\mu \partial x_\mu} + A(r^2)x_\mu \frac{\partial \Psi}{\partial x_\mu} + C(r^2)\Psi = 0 \,, \quad (\mu = 1, 2, 3) \tag{1.1}$$

where $A(r^2)$, $C(r^2)$ are analytic functions of the real variable $r^2 = x_\mu x_\mu$ $[\mu = 1, 2, 3]$ (Repeated indices mean the summation convention is used.) In this paper we shall investigate the four variable analogue of this equation, $T_4[\Psi] = 0$, and show that many of Bergman's results carry over to this case. Here, we need in many instances, the methods of several complex variables in order to find the natural generalizations.

In Bergman's theory,[1] the integral operator $B_3\ [f]$ plays an important role in studying the solutions of (1.1). In our case, there is an analogous operator [7]-[10], which is a four-variable analogue to $B_3\ [f]$:

$$H[X] = B_4[f] \equiv -\frac{1}{4\pi^2}\iint_D \frac{d\eta}{\eta}\frac{d\xi}{\xi} f(u; \eta, \xi) \,, \quad X \equiv (x_1, x_2, x_3, x_4) \,,$$

$$u \equiv x_1\Big(1 + \frac{1}{\eta\xi}\Big) + ix_2\Big(1 - \frac{1}{\eta\xi}\Big) + x_3\Big(\frac{1}{\xi} - \frac{1}{\eta}\Big) + ix_4\Big(\frac{1}{\xi} + \frac{1}{\eta}\Big) \tag{1.2}$$

and $D = \{|\xi| = 1\} \times \{|\eta| = 1\}$. The operator $B_4\ [f]$ maps analytic functions of three-complex variable onto harmonic, function-elements of four-variables. One may realize how analytic functions are transformed into harmonic functions, by considering the powers of u, which act as generating functions for the homogeneous, harmonic polynomials, $H_n^{kl}[X]$,

$$u^n = \sum_{k,l=0}^{n} H_n^{kl}[X]\xi^{-k}\eta^{-l} \tag{1.3}$$

The polynomials $H_n^{kl}(X)$ $[k, l = 0, 1, \cdots, n;\ n = 0, 1, 2, \cdots]$ form a complete, linearly independent system [13] [7] [8]. From the Cauchy formula for two complex variable we have an integral representation for these polynominals given by,

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[1] See Bergman [1], Chapter II.