MEASURABLE SETS OF MEASURES

LESTER DUBINS AND DAVID FREEDMAN

1. Introduction. Let M be the set of all countably additive, finite, signed measures on a σ-field Σ of subsets of a set X. There is a natural definition of measurability in M, namely, a subset of M is *measurable* if it is an element of Σ^*, the smallest σ-field of subsets of M such that: for each $A \in \Sigma$ the function $\mu \to \mu(A)$ is measurable from M to the Borel line. The purpose of this note, motivated by questions arising from (Dubins and Freedman, 1963) is to investigate the measurability and category of interesting subsets of M, under the assumption that Σ is countably generated.

Here are some results. If X is compact metric, and Σ is the σ-field of Borel subsets of X, then any subset of X with the Baire property is measurable for a residual set of probability measures (3.17). If also X is uncountable, there are weakly open, but not Σ^*-measurable, subsets of M; see (3.2). There is a G_δ in the three-dimensional unit cube whose convex hull is not Borel (3.22). If F is a continuous, strictly monotone, purely singular distribution function on the unit interval, then F is differentiable only on a set of the first category (4.8).

2. The abstract case. Let X be a nonempty set, $\mathscr{F}$ a countable field of subsets of X, and Σ the smallest σ-field including $\mathscr{F}$.

2.1. *Let $\mathscr{A}$ be a σ-field of subsets of a set Ω, and let φ map Ω into M. Then φ is measurable from $(\Omega, \mathscr{A})$ to (M, Σ^*) if and only if the function $\omega \to \varphi(\omega)(A)$ is measurable from $(\Omega, \mathscr{A})$ to the Borel line for each $A \in \mathscr{F}$.*

Proof. Routine.

2.2. *If φ is a measurable map from $(\Omega, \mathscr{A})$ to (M, Σ^*), and f is a bounded, measurable function from $(\Omega \times X, \mathscr{A} \times \Sigma)$ to the Borel line, then $\omega \to \int_X f(\omega, x)\varphi(\omega)(dx)$ is a measurable function from $(\Omega, \mathscr{A})$ to the Borel line.*

Proof. Extend from indicators of measurable rectangles.

2.3. *The σ-field Σ^* is countably generated.*

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