RINGS OF ARITHMETIC FUNCTIONS

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1. Introduction. Let F denote a fixed but arbitrary field and let Z denote the set of positive integers. By an *arithmetic function* f is meant a function from Z to F, that is to say $f(n) \in F$ for all $n \in Z$. If f, g are two arithmetic functions, the sum $h = f + g$ is defined by means of

$$h(n) = f(n) + g(n) \qquad (n \in Z) . \tag{1}$$

There are two products that are of interest, the *ordinary* product defined by

$$h(n) = f(n)g(n) \qquad (n \in Z) , \tag{2}$$

and the Dirichlet product defined by

$$h(n) = \sum_{rs=n} f(r)g(s) \qquad (n \in Z) , \tag{3}$$

where the summation on the right is extended over all factorizations $rs = n$. We shall denote the ordinary product by $f \circ g$ and the Dirichlet product by $f * g$.

Let S denote the set of arithmetic functions as defined above. It is well known and easy to prove that the system

$$\Omega = (S, f, \circ) \tag{4}$$

is a commutative ring. The multiplicative identity of Ω is defined by

$$v(n) = 1 \qquad (n \in Z) . \tag{5}$$

Clearly Ω is not a domain of integrity; note however that there are no nilpotent elements in Ω. On the other hand the system

$$\Delta = (S, f, *) \tag{6}$$

is a domain of integrity. The multiplicative identity of Δ is given by

$$u(n) = \begin{cases} 1 & (n = 1) \\ 0 & (n > 1) . \end{cases} \tag{7}$$

Moreover the function f has an inverse (relative to $*$) if and only if

$$f(1) \neq 0 ; \tag{8}$$

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