FUNCTIONS WITH CONVEX MEANS

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1. Introduction. Let f be Lebesgue summable on $[0, a]$, $a > 0$. The function F_1 defined on $[0, a]$ by $F_1(x) = 1/x\int_0^x f(t)dt$, $F(0) = f(0)$ is called the mean of the function f. Inductively, we may define the Nth mean of f on $[0, a]$ by $F_N(x) = 1/x\int_0^x F_{N-1}(t)dt$, $F_N(0) = f(0)$, provided, of course, that F_{N-1} is summable on $[0, a]$. Some questions involving the mean of certain classes of functions have been examined in Beckenbach [1] and Bruckner and Ostrow [2].

The primary purpose of this paper is to consider the problem of determining when a function f has its Nth mean convex for N sufficiently large. To develop the necessary machinery, we devote § 2 to obtaining some properties of the means which we shall need in the sequel, and we devote § 3 to obtaining representations for functions f possessing means of all orders. In particular, Theorem 3 shows how a wide class of functions f admit representations as sums of infinite series whose Nth term involves the Nth mean of f. Then in § 4 we examine the question posed at the beginning of this paragraph. (Lemma 6 together with Theorem 5 yields a condition which is necessary and sufficient for a sufficiently well behaved function to have its Nth mean convex. We then use this theorem to obtain two sufficient conditions for a starshaped function to have its Nth mean convex for N sufficiently large. This is done in Theorems 6 and 7. Finally, in Theorem 8, we use the Baire Category Theorem to show that not every continuous starshaped function has one of its means convex.

2. Preliminaries. We begin with some remarks and simple observations. In the sequel, we shall denote the Nth mean of f by $F_N(f:x)$. As circumstances warrant, this notation will be shortened to $F_N(f)$, $F_N(x)$, or simply F_N.

The fact that f is summable does not insure that means of all orders exist. Thus, the mean of the function f given by $f(x) = x^{-1}(\log x)^{-2}, f(0) = 0$ is $F_1(x) = -x^{-1}(\log x)^{-1}, F_1(0) = 0$, but F_1 is not summable on any neighborhood of the origin.

DEFINITION. The function f is in the class $M(a)$ provided f possesses means of all orders on $[0, a]$.

It is easily seen from Lemma 1 below that $f \in M(a)$ if the functions

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