PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 1, 1965

CHEBYSHEV APPROXIMATION TO ZERO

JAMES M. SLOSS

In this paper we shall be concerned with the questions of existence, uniqueness and constructability of those polynomials in $k+1$ variables $(x_1, x_2, \cdots, x_k, y)$ of degree not greater than n_s in x_s and m in y which best approximate zero on $I_1 \times I_2 \times \cdots \times I_{k+1}, I_s = [-1, 1]$, in the Chebyshev sense.

It is a classic result that among all monic polynomials of degree not greater than n there is a unique polynomial whose maximum over the interval $[-1, 1]$ is less than the maximum over $[-1, 1]$ of any other polynomial of the same type and moreover it is given by $\tilde{T}_n(x) = 2^{1-n} \cos [n \text{ are } \cos x]$, the normalized Chebyshev polynomial.

Our method of attack will be to prove a generalization of an inequality for monic polynomials in one variable concerning the lower bound of the maximum viz. $\max_{-1 \leqq x \leqq 1} | P_n(x) | \geqq 2^{1-n}$ where $P_n(x)$ is a monic polynomial of degree not greater than n. The theorem will show that the only hope for uniqueness is to normalize our class of polynomials. This is done in a very natural way viz. by considering only polynomials, if they exist, of the form:

$$(0.1) \qquad P(x_1, x_2, \cdots, x_k, y) = A_m(x_1, \cdots, x_k)y^m + A_{m-1}(\cdots)y^{m-1} + \cdots + A_0(\cdots)$$

for which $A_m(x_1, x_2, \cdots, x_k)$ is the best polynomial approximation to zero on $I_1 \times I_2 \times \cdots \times I_k$. Thus if $k = 1$, we consider only polynomials of the form:

$$(0.2) \qquad P(x_1, y) = \tilde{T}_n(x_1)y^m + A_{m-1}(x_1)y^{m-1} + \cdots + A_0(x_1) .$$

We find in the case of (0.2) that there is a unique best polynomial approximation and it is given by $\tilde{T}_n(x_1)\tilde{T}_m(y)$. Thus we can consider the question of existence, uniqueness and constructability of a polynomial of the form:

$$(0.3) \qquad P(x_1, x_2, y) = \tilde{T}_{n_1}(x_1)\tilde{T}_{n_2}(x_2)y^m + A_{m-1}(x_1, x_2)y^{m-1} + \cdots + A_0(x_1, x_2)$$

that best approximates zero. We find in this case there is a unique best polynomial approximation and it is given by $\tilde{T}_{n_1}(x_1)\tilde{T}_{n_2}(x_2)\tilde{T}_m(y)$. Continuing in this way we shall show that the question is meaning-

Received December 18, 1963.