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FINITISTIC GLOBAL DIMENSION FOR RINGS

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The finitistic global dimensions $lfPD(R), lFPD(R)$, and $lFID(R)$ are defined for a ring R. We obtain the following results for R semiprimary with Jacobson radical N. C is a simple left R-module and $l.\dim_R C < \infty$, and suppose that $l.\dim_R N^{i-1}/N^i < \infty$ for $i \geqq 3$. Then $m \leqq lfPD(R) = lFPD(R) \leqq (m+1)$. Theorem 2: Suppose that $l.\operatorname{inj.}\dim_R P \leqq l.\operatorname{inj.}\dim_R R/N^2 < \infty$ for every projective (R/N^2)-module P and that $l.\operatorname{inj.}\dim_R N^{i-1}*N^i < \infty$ for $i \geqq 3$. Then $lFID(R) = l.\operatorname{inj.}\dim_R R < \infty$. The method of proof uses a result of Eilenberg and a result of Bass on direct limits of modules together with the lemma: If M is a left R-module such that $N^{k-1}M \neq 0$ and $N^k M = 0$, then every simple direct summand of N^{k-1} is isomorphic to a direct summand of N^{k-1}/N^k.

1. We begin by discussing some further properties of perfect and left perfect rings. The rest of the paper is devoted to giving sufficient conditions for finiteness and equality of certain finitistic global dimensions for a semi-primary ring.

Let R be a ring (with identity). By an R-module we shall always mean a left unitary module over R. In ([7]) and ([10]), Eilenberg and Nakayama define what they called minimal epimorphisms. Bass ([1]) altered this definition to call minimal epimorphisms projective covers. Eilenberg ([7)] studied the dimension theory for modules having minimal epimorphisms and said that a category of modules is perfect if every member of the category has a projective cover. Thus Bass ([1]) called a ring R for which every R-module has a projective cover a left perfect ring. There are two special types of left perfect rings about which we are particularly interested. One is the semi-primary ring R where the Jacobson radical (J-radical) N is nilpotent and R/N is semi-simple with minimum condition (semi-simple), and the other is a ring with minimum condition on left ideals (left Artinian ring).

We define the following finitistic global dimensions for R, using the definitions and notation of ([1]) and ([3]). $lFPD(R) = \sup l.\dim_R M$ for all R-modules of finite projective dimension, $lfPD(R) = \sup l.\dim_R M$ for all finitely generated (f.g.) R-modules of finite projective dimension, $lFWD(R) = \sup w.l.\dim_R M$ for all R-modules of finite weak dimension, $lFID(R) = \sup l.\operatorname{inj.}\dim_R M$ for all R-modules of finite injective dimension.

In § 2 we discuss some further properties of left perfect and perfect rings.

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