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PROPER ORDERED INVERSE SEMIGROUPS

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Let S be an ordered inverse semigroup, that is, an inverse semigroup with a simple order $<$ which satisfies the condition:

$$x < y \text{ implies } xz \leqq yz \text{ and } zx \leqq zy .$$

Let E be the subsemigroup of S constituted by all the idempotents of S. By a result of Munn, $\Gamma = S/\sigma$ is an ordered group, where σ is the congruence relation such that $x\sigma y$ if and only if $ex = ey$ for some $e \in E$. An ordered inverse semigroup S is called proper if the σ-class I which is the identity element of Γ contains only idempotents of S.

In a proper ordered inverse semigroup S, let $\Gamma(e)(e \in E)$ be the set of those members of Γ which intersect nontrivially with R_e. Each element of S can be represented in the form (α, e), where $e \in E$ and $\alpha \in \Gamma(e)$. We define $e^\alpha = a^{-1}a \in E$, where $a = (\alpha, e)$. Then $\Gamma(e)$ and e^α satisfy the following six conditions:

(i) $\bigcup_{e \in E} \Gamma(e) = \Gamma$;

(ii) $I \in \Gamma(e)$ **and** $e^I = e$;

(iii) if $f \leqq e$ in the semilattice with respect to the natural ordering of the commutative idempotent semigroup E and $\alpha \in \Gamma(e)$, then $\alpha \in \Gamma(f)$ and $f^\alpha \leqq e^\alpha$ in the semilattice E;

(iv) if $\alpha \in \Gamma(e)$ and $\beta \in \Gamma(e^\alpha)$, then $\alpha\beta \in \Gamma(e)$ and $e^{\alpha\beta} = (e^\alpha)^\beta$;

(v) if $\alpha \in \Gamma(e)$, then $\alpha^{-1} \in \Gamma(e^\alpha)$;

(vi) if $\alpha \in \Gamma(e) \cap \Gamma(f)$ and $e \leqq f$, then $e^\alpha \leqq f^\alpha$.

Also the product and the order in S determined by

$(\alpha, e)(\beta, f) = (\alpha\beta, (e^\alpha f)^{\alpha^{-1}})$;

$(\alpha, e) \leqq (\beta, f)$ **if and only if either** $\alpha < \beta$ **or** $\alpha = \beta, e \leqq f$.

Next we prove conversely a theorem asserting that, for an ordered commutative idempotent semigroup E and an ordered group Γ, if $\Gamma(e)$ and e^α satisfy the six conditions above, then the set $\{(\alpha, e); e \in E, \alpha \in \Gamma(e)\}$ is a proper ordered inverse semigroup with respect to the product and the order mentioned above. Besides this, we present other characterizations of special cases.

Ordered semigroups were studied systematically in [4], [5], [6]. In [4], we studied ordered idempotent semigroups. In an ordered semigroup, the set of all the idempotents always constitutes a subsemigroup and so the study of ordered idempotent semigroups will clarify the structure of this subsemigroup. In [5], we were essentially concerned with ordered regular semigroups. As the first step of the study of these semigroups, in that paper we determined all the types

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