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A GENERALIZATION OF THE COSET DECOMPOSITION OF A FINITE GROUP

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Let G be a finite group, and suppose that G is partitioned into disjoint subsets: $G = \bigcup_{i=1}^{h} A_i$. If the A_i are the left (or right) cosets of a subgroup $H \subseteq G$, then the products xy, where $x \in A_i$ and $y \in A_j$, represent all elements of any A_k the same number of times. It turns out that certain other decompositions of G of interest in algebra enjoy this same property; we will call such a partition π an α-partition.

In this paper all α-partitions are determined in the case G is a cyclic group of prime order p; they arise by choosing a divisor d of $p-1$, and letting the A_i be the sets on which the d'th power residue symbol $(x/p)_d$ has a fixed value. It is shown that if an α-partition is invariant under the inner automorphisms of G (strongly normal) then it is also invariant under the antiautomorphism $x \rightarrow x^{-1}$. For such α-partitions (called weakly normal) it is shown that the set A_i containing the identity element is a group. An example shows that this need not hold for nonnormal partitions.

1. For any α-partition π, let N_{ijk} denote the number of times each element of A_k is represented among the products xy, $x \in A_i$, $y \in A_j$. Then if $\mathfrak{A}(G)$ is the group algebra of G over a field K, and if we put

$$a_i = \sum_{x \in A_i} x \, , \tag{1}$$

it is clear that $a_i a_j = \sum_{k=1}^{h} N_{ijk} a_k$. Therefore the vector space spanned over K by $a_1, \cdots, a_h$ is a subalgebra $\mathfrak{A}_\pi$ of $\mathfrak{A}(G)$, with structure constants N_{ijk}. Conversely, if $\pi : G = \bigcup_{i=1}^{h} A_i$ is any partition of G into disjoint subsets, and if the elements a_i defined by (1) span a subalgebra of $\mathfrak{A}(G)$, then π is an α-partition.

In the case where π is the decomposition of G into the cosets of a normal subgroup H whose order m is not divisible by the characteristic of K, the algebra $\mathfrak{A}_\pi$ is the group algebra $\mathfrak{A}(G/H)$ of the factor group G/H. For then the elements a_i/m form a group isomorphic to G/H, and are a basis of $\mathfrak{A}_\pi$.

In this paper some of the elementary properties of α-partitions are developed. I plan in a second paper to discuss in more detail the structure of the algebras $\mathfrak{A}_\pi$ and their application to the representation of G by matrices.

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