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SIMPLE QUADRATURES IN THE COMPLEX PLANE

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Given a class S of functions that are Riemann integrable on $[0, 1]$. A quadrature formula $\int_0^1 f(x)dx = \sum_{i=1}^{\infty} a_i f(x_i)$ is called a *simple quadrature for* S if the x_i are distinct and if both the a_i and the x_i are fixed and independent of the particular function of S selected. It is known that if S is too large, for example if $S = C[0, 1]$, a simple quadrature cannot exist. On the other hand, if S is sufficiently restricted, for example the class of all polynomials, then simple quadratures exist.

The present paper investigates further the existence of simple quadratures. It is proved among other things that if S is the class of analytic functions that are regular in the closure of an ellipse with foci at ± 1, a simple quadrature exists for the weighted integral $\int_{-1}^{+1} (1-x^2)^{1/2} f(x)dx$ provided we allow the abscissas x_i to take on complex values.

1. Simple Quadratures. In [3], the author studied the following question. Suppose that there has been given a fairly extensive class S of real functions that are Riemann integrable on $[0, 1]$. Does there exist a quadrature formula of the form

$$\int_0^1 f(x)dx = \sum_{i=1}^{\infty} a_i f(x_i) \tag{1}$$

which is valid for all functions of the class S? The abscissas x_i are assumed distinct, and both the weights a_i and the abscissas x_i are fixed and independent of the particular function of S selected. A quadrature of the form (1) was called a *simple* quadrature to contrast it with quadratures of the form

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} a_{in} f(x_{in}) \tag{2}$$

which allow more freedom than (1) and have accordingly been more frequently investigated. See, e.g., Szegö [9], Chap. 15.

In [3], we found, broadly speaking, that if S is fairly small, simple quadratures exist, while if S has too many functions in it, simple quadratures do not exist. Thus, for instance, there exists a simple quadratures for the class of all polynomials (of unbounded degree), while there cannot exist a simple quadrature for the class of continuous functions. See also Davis [7], Chap. 14, where this question is treated in the framework of weak* convergence.

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