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FAMILIES OF PARALLELS ASSOCIATED WITH SETS

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There exist sets S in Euclidean space E_n which have an interesting association with a family $\mathscr{P}$ of parallel lines. For instance S and $\mathscr{P}$ may be so related that each point of S lies on a member of $\mathscr{P}$ which intersects S in either a line segment or a point. There exist compact sets $S \subset E_2$ such that every finite collection of points in S is contained in some collection of parallel lines each of which intersects S in a single point, and yet no infinite family $\mathscr{P}$ of parallel lines exists having the same property and covering S. This paper contains a theorem which enables one to determine the existence of a family of parallel lines each of which intersects S in a line segment or point and which as a family covers S.

Secondly we show that the points, the closed line segments, the closed convex triangular regions, and the closed convex sets bounded by parallelograms are the only compact convex sets B in E_2 which have the following property. If A is a closed connected set disjoint from B and if every 3 or fewer points of A lie on parallel lines intersecting B, then A is covered by a family of parallel lines each of which intersects B.

Finally, we obtain a theorem of Krasnoselskii type. Intuitively, this may be stated as follows. Suppose S is a compact set in E_n and suppose there exists a plane H such that every n points of S can see H via S along parallel lines. Then all the points of S can see H via S along parallel lines.

The above results appear in Theorems 3, 2, 1 in that order. The appendix at the end contains the theorems of Helly, Krasnosel'skii and other results used. Furthermore, the reader is recommended to consult the compendium "Helly's theorem and its relatives" by Danzer, Klee and Grünbaum [1]. In order to proceed logically we adopt the following notations.

NOTATION. If S is a set in n-dimensional Euclidean space E_n, then closure of $S = \mathrm{cl}\, S$, interior of $S = \mathrm{int}\, S$, boundary of $S = bd\, S$, convex hull of $S = \mathrm{conv}\, S$. If $x \in E_n$, $y \in E_n$, $x \neq y$, then $L(x, y) =$ line containing x and y, $xy =$ closed segment joining x and y, $\mathrm{intv}\, xy =$ relative interior of the segment xy, $R(x, y) =$ ray having x as endpoint and containing y. The empty set is indicated by 0 and the origin of E_n by $\varnothing$. Set union, intersection and difference are denoted by $\cup$, $\cap$ and $\sim$ respectively.

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