PACIFIC JOURNAL OF MATHEMATICS
Vol. 16, No. 1, 1966

ON THE DETERMINATION OF CONFORMAL IMBEDDING

TILLA KLOTZ

Two imbedding fundamental forms determine (up to motions) the smooth imbedding of an oriented surface in E^3. The situation is, however, substantially different for the sufficiently smooth conformal imbedding of a Riemann surface R in E^3. Conventionally such an imbedding is achieved by a conformal correspondence between R and the Riemann surface R_1 determined on a smoothly imbedded oriented surface S in E^3 by its first fundamental from I. We show that except where $H \cdot K = 0$ on S, such an R_1 conformal imbedding of R in E^3 is determined (up to motions) by the second fundamental form II on S, expressed as a form on R. In particular, I is determined by II on R_1, where $H \cdot K \neq 0$ on S.

Similar remarks are valid for two less standard methods of conformal imbedding. If an oriented surface S is smoothly imbedded in E^3 so that $H > 0$ and $K > 0$, then II defines a Riemann surface R_2 on S. And, if S is imbedded so that $K < 0$, then II' given by

$$H'II' = KI - HII$$

with

$$H' = -\sqrt{H^2 - K}$$

defines a Riemann surface R_2' on S. Thus a conformal correspondence between R and R_2 (or R_2') is called an R_2 (or R_2') conformal imbedding of R in E^3. We show that I on S, expressed as a form on R, determines the R_2 or (wherever $H \neq 0$ and sign H is know) the R_2' imbedding of R in E^3 (up to motions). In particular, I determines II on R_2, and (where $H \neq 0$, and sign H is known) on R_2' as well. Finally, we give restatements of the fundamental theorem of surface theory in forms appropriate to R_1, R_2 and R_2' conformal imbeddings in E^3.

The two fundamental forms which determine (up to motions) the smooth imbedding of an oriented surface in E^3 are, of course, related by various equations. But neither form determines the other, except in very special cases. Thus, for instance, isometric imbeddings of a surface in E^3 may differ essentially unless (to cite a famous example) the surface is compact, and the common metric imposed by imbedding has positive Gaussian curvature.

2. Consider an oriented surface S which is C^3 imbedded in E^3.

Received January 10, 1964. This research was supported under NSF grant GP-1184.